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MULTIPLIERS FOR THE ORLICZ SPACES L_A

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Let m be a bounded measurable function on R^n. One can define a linear transform T_m, whose domain is $L^2 \cap L_A$, by the following relation between Fourier transforms

$$\widehat{(T_m f)}(x) = m(x)\hat{f}(x), \quad f \in L^2 \cap L_A,$$

where L_A is an Orlicz space.

We shall say that m is a *multiplier for* L_A if whenever $f \in L^2 \cap L_A$ then $T_m f$ is also in L_A, and T_m is bounded, that is

$$(*) \qquad \|T_m f\|_A \leqq C\|f\|_A, \ f \in L^2 \cap L_A$$

(with C independent to f).

The smallest C for which $(*)$ holds will be called the *norm* of the multiplier. We denote $\mathfrak{M}_A$ the class of multipliers with the indicated norm.

EXAMPLE: If $A(t) = \frac{|t|^2}{2}$, then $\mathfrak{M}_A = \mathfrak{M}_{L^2}$ is the class of all bounded measurable functions and the multiplier norm is identical with the L^∞-norm.

It is well known that if $A_p(t) = \frac{|t|^p}{p}$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then $L_{A_p} = L^p$ and $\mathfrak{M}_{L^p} = \mathfrak{M}_{L^{p'}}$ [2]. We will generalize this fact in the Orlicz space L_A for any N-function A.

If A and $\tilde{A}$ are complementary N-functions, a gereralized version of Hölder's inequality

$$|u(x)v(x)dx| \leqq 2\|u\|_A\|v\|_{\tilde{A}}, \quad u \in L_A, \quad v \in L_{\tilde{A}}$$

can be obtained by applying Young's inequality to

$$|u(x)|/\|u\|_A \text{ and } |v(x)|/\|v\|_{\tilde{A}} \text{ [1].}$$

THEOREM. *Suppose A and $\tilde{A}$ are complementary N-functions, then*

$$\mathfrak{M}_A = \mathfrak{M}_{\tilde{A}}.$$

Proof. Let σ denote the involution $\sigma(f)(x) = \bar{f}(-x)$. As is immediately verified $\sigma^{-1}T_m\sigma = T_{\bar{m}}$, therefore if m belongs to $\mathfrak{M}_A$, so does $\bar{m}$; moreover $\bar{m}$

has the same norm of m. Now by Plancherel's formula,

$$\int T_m f\bar{g}dx=\int m(x)\hat{f}(x)\overline{\hat{g}(x)}dx=\int \hat{f}(x)\overline{\bar{m}(x)\hat{g}(x)}dx=\int f\overline{T_{\bar{m}}g}dx,$$

whenever $f, g\in L^2$. Assume in addition that $f\in L_{\tilde{A}}$, $g\in L_A$ and $\|g\|_A\leqq 1$. Then

$$|\int T_m f\bar{g}dx|\leqq 2\|f\|_{\tilde{A}}\|T_{\bar{m}}g\|_A\leqq 2C\|f\|_{\tilde{A}},$$

where C is the norm of the multiplier m (or $\bar{m}$) in $\mathfrak{M}_A$. We may assume that $T_m f\neq 0$ in $L_{\tilde{A}}$ so that

$$K=\|L_{T_m f}\|>0,$$

where $L_{T_m f}(\bar{g})=\int T_m f\bar{g}dx$ and $\|\cdot\|$ is the norm in the dual space L'_A.

Taking the supremum over all indicated g, gives

$$\|L_{T_m f}\|\leqq 2C\|f\|_{\tilde{A}}.$$

On the other hand, let

$$\bar{g}(x)=\begin{cases}\tilde{A}\left(\frac{|T_m f(x)|}{K}\right)/\frac{T_m f(x)}{K} & \text{if } T_m f(x)\neq 0\\ 0 & \text{if } T_m f(x)=0\end{cases}$$

If $\|\bar{g}\|_A>1$, then for sufficiently small $\varepsilon>0$, we have

$$\frac{1}{\|\bar{g}\|_A-\varepsilon}\int A(|\bar{g}(x)|)dx\geqq\int A\left(\frac{|\bar{g}(x)|}{\|\bar{g}\|_A-\varepsilon}\right)dx>1$$

Letting $\varepsilon\to 0^+$ we obtain, using the inequality $A(\tilde{A}(t)/t)<\tilde{A}(t)$, $t>0$ [1],

$$\|\bar{g}\|_A\leqq\int A(|\bar{g}(x)|)dx=\int A\left[\tilde{A}\left(\frac{|T_m f(x)|}{K}\right)/\frac{T_m f(x)}{K}\right]dx$$

$$<\tilde{A}\left(\frac{|T_m f(x)|}{K}\right)dx=\frac{1}{\|L_{T_m f}\|}\int|T_m f\,\bar{g}|dx\leqq\|\bar{g}\|_A.$$

This contradiction shows that $\|\bar{g}\|_A\leqq 1$. Now

$$\|L_{T_m f}\|=\sup_{\|\bar{g}\|_A\leqq 1}|L_{T_m f}(\bar{g})|\geqq\|L_{T_m f}\|\int\tilde{A}\left(\frac{|T_m f(x)|}{\|L_{T_m f}\|}\right)dx$$

so that

$$\int\tilde{A}\left(\frac{|T_m f(x)}{|\|L_{T_m f}\|}\right)dx\leqq 1.$$

Thus $\|T_m f\|_{\tilde{A}} \leqq \|L_{T_m f}\| \leqq 2C\|f\|_{\tilde{A}}$ and $m \in \mathfrak{M}_{\tilde{A}}$. Since the situation is symmetric in L_A and $L_{\tilde{A}}$, we have $\mathfrak{M}_A = \mathfrak{M}_{\tilde{A}}$. Q. E. D.

References

1. Adams; *Sobolev spaces*, Academic Press, 1975.
2. E. M. Stein; *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.

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