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COMPACTNESS OF HOMOGENEOUS SPACES WITH FINITE VOLUMES

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Let G be a locally compact group and H be a closed subgroup such that G/H admits a finite G–invariant measure. Then under suitable restrictions on G or H it ensures G/H to be compact. For example, such is the case when G is a connected Lie group and H is any closed subgroup with finitely may connected components [4]. Also K. C. Sit generalized the above Mostow's result and proved [5] that G/H is compact whenever G is a locally compact and σ–compact group with the open identity component and H is the fixed points of a set of automorphisms and G/H admits a fiinite invariant measure.

In this paper we prove the following;

THOEREM. *Let G be a $[C]$–group and H the centralizer of an element of x of G such that $\mathcal{D}=\{gxg^{-1} : g\in G\}$ is closed. If G/H admits a finite invariant measure, then G/H is compact.*

A simple example provides a $[C]$–group G with non–open identity component; $G=H\times K$ where H is any connected locally compact group and K is a compact, non–discrete and totally disconnected locally compact group.

§ 1. Preliminary lemmas.

For a locally compact group G, G_0 will denote the identity component of G and the group G will be called a $[C]$–group if the quotient group G/G_0 is compact. It is well known that a $[C]$–group G can be approximated by a Lie groups; each neighbourhood of the identity contains a compact normal subgroup K of G such that G/K is a Lie group.

We shall modify this well known approximation theorem so that we can apply directly in proving our theorem.

We know that each neighbourhood of the identity of a compactly generated locally compact group G containes a compact normal subgroup H such that the quotient group G/H satisfies the second axiom of countability [1, p 71]. In particular, this is true for $[C]$–groups. Now let K (resp. H) be a compact normal subgroup of a $[C]$–group such that G/K is a Lie group (resp. G/H satisfies the second axiom of countability). Then HK is acompact normal subgroup and the second countable group G/HK is isomorphic (topologically) to $(G/K)/(HK/K)$ which is a Lie group. Thus we have

LEMMA 1. *A [C]-group can be approximated by second countable Lie group.*

A locally compact space X is called a homogeneous G-space if G acts on X transitively. Thus G/H is a homogeneous G-space for any closed subgro up H by a left translation. A regular Borel measure μ on X is G-invariant if $\mu(gE)=\mu(E)$ for each Borel measurable set E and $g\in G$.

The following lemma is proved by Greenleaf, Moskowitz and Rothschild [2, p. 151].

LEMMA 2. *Let G be a second countable Lie group and A be the fixed points of a set of automorphisms of G. If G/A admits a G-invariant measure, then G/A is compact.*

From now on a locally compact group will be assumed to be σ-compact unless otherwise specified. Let G (resp. G') be a locally compact group and let X (resp. X') be a homogeneous G-space (resp. G'-space).

LEMMA 3. *If $\pi : G\rightarrow G'$ is an open and continuous epimorphism and $\eta : X\rightarrow X'$ an equivariant continuous surjection, then η is an open mapping and a finite G-invariant measure μ on X can be transformed into a finite G'-invariant measure μ' on X'.*

Although this lemma is well known, we sketch the proof for the convenience sake.

The proof of the openness of η is based on the fact that a continuous surjection from a locally compact and σ-compact group to a Baire space (which is also a G-space) is open [3, p. 39]. Applying this fact to the mapping

$$f : G'\rightarrow X';\ g\rightarrow g\cdot\eta(x),$$

we see that f is open. Now the equality, for any neighbourhood V of the identity of G, $\pi(V)\eta(x)=\eta(Vx)$ proves that η is open [5]. We show that μ', defined on X' by $\mu'(E)=\mu(\eta^{-1}(E))$ for every Borel set, is a regular measure. The μ' is clearly a measure. Since μ' is finite on X, it suffices to show that

$$\mu'(E)=\sup\{\mu'(K) : K' \text{ is compact, } K'\subset E\}$$

for each measurable set E in X'. Clearly we have

$$\mu'(E)\geq \sup\{\mu'(K') : K' \text{ is compact, } K'\subset E\}\geq$$

$$\geq \sup\{\mu(\eta^{-1}(K')) : K' \text{ is compact, } \eta^{-1}(K') \subset \eta^{-1}(E)\}.$$

For a compact set $K \subset \eta^{-1}(E)$, $\eta(K)$ is compact and $\mu(K) \leq \mu(\eta^{-1}(\eta(K)))$ and it follows that

$$\sup\{\mu(\eta^{-1}(K')) : \eta^{-1}(K') \subset \eta^{-1}(E),\ K' \text{ is compact}\}$$
$$\geq \sup\{\mu(K) : K \text{ is compact, } K \subset \eta^{-1}(E)\}.$$

The second term of the inequality is, by the regularity of μ, $\mu\eta(^{-1}(E))$ and this is $\mu'(E)$ by the definition of μ'. Thus we have shown that $u'(E) = \sup \{\mu'(K') : K' \text{ is compact, } K' \subset E\}$, the regularity of μ'. The G'-invariance of μ follows from the fact that η is an equivariant.

LEMMA 4. [4, Lemma 2.5] *Let $H \subset F$ be closed subgroups such that G/H admits a finite G-invariant measure μ. Then G/F and F/H admits, respectively, finite G-invariant and F-invariant measures of which μ is a product.*

§2. The Proof of Theorem

By Lemma 1, there is a compact normal subgroup K such that G/K is a second countable Lie group. Since $KH \supset K$ and KH is closed, G/KH admits a finite invariant measure (Lemm 4). Since $(G/H)/(KH/H)$ is homeomorphic to G/KH and KH/H is compact, G/H is compact if and only if G/KH is compact. Thus we reduced the problem to "whether G/KH is compact provided G/KH admits a finite G-invariant measure".

Let λ and λ' be the usual actions of G on G/HK and G/K on $(G/K)/(KH/K)$, respectively. Then as the diagram shown below, there corresponds a continuous surjection (in fact, a homeomorphism) $\eta : G/KH \to (G/K)/(KH/K)$ defined by $\eta : yHK \to yK(HK/K)$ so that the diagram commutes, i. e., η is an equivariant mapping.

$$\begin{array}{ccc} G \times G/KH & \xrightarrow{\lambda} & G/HK \\ \downarrow{\scriptstyle \pi\times\eta} & & \downarrow{\scriptstyle \eta} \\ G/K \times ((G/K)/(KH/K)) & \xrightarrow{\lambda'} & (G/K)/(KH/K) \end{array}$$

In the diagram, π denotes the canonical projection of G onto G/K. Therefore, by Lemma 2 the G-invariant finite measure on G/KH induces a finite G/K-invariant measure on $(G/K)/(KH/K)$.

Let H' be the centralizer of xK in G/K. Since $\pi^{-1}(H') = \{g \in G : g^{-1}x^{-1}gx \in K\}$ and contains H, $HK/K \subset H'$ and $H'/(HK/K)$ admit finite invariant measures.

Since G/K is second countable Lie group; we can apply Lemma 2 and deduce that $(G/K)/H$ is compact.

Note that $(G/K)/H'$ is homeomorphic to $((G/K)/(KH/K))/(H'/(KH/K))$. Therefore, the compactness of G/HK (which is homeomorphic to $(G/H)/(KH/K)$) follows from the compactness of $H'/(KH/K)$ which remains to be shown.

Since $H'/(KH/K) \simeq \pi^{-1}(H')/KH$ is a continuous image of $\pi^{-1}(H')/H$, it suffices to show that $\pi^{-1}(H')/H$ is compact. Consider a continuous map j_x on G defined by $j_x(g) = g(x)g^{-1}$, $g \in G$. Then $j_x^{-1}(xK) = \pi^{-1}(H')$ and $j_x^{-1}(x) = H$. Therefore the restriction f of j_x to $\pi^{-1}(H')$ is continuous on $\pi^{-1}(H')$ which is locally compact and σ-compact and the image of f is a compact set $\mathcal{D} \cap xK$.

We shall show that f is an open mapping. Since every element of $X = \mathcal{D} \cap xK$ can be written as gxg^{-1} for some g in $\pi^{-1}(H')$, the group $\pi^{-1}(H')$ acts on X by conjugation. In fact let $g' \in \pi^{-1}(H')$ then, because K is normal, $g'(gxg^{-1})g'^{-1} \subset g'(xK)g'^{-1} \subset g'xg'^{-1}K \subset xK$. Moreover $\pi^{-1}(H')$ acts transitively on X. To see this let z and z' be any two elements in X and write $z = xk$ and $z' = xk'$ $(k, k' \in K)$. Clearly there exists an element h in H such that $hkh^{-1} = k'$ and we have $hzh^{-1} = (hxh^{-1})(hkh^{-1}) = xh' = z'$, proving $\pi^{-1}(H')$ acts on X transitively. Thus f is a continuous map of a locally compact and σ-compact group $\pi^{-1}(H')$ onto a Bair homogeneous G-space X; f is an open mapping (see the proof of Lemma 3).

Since $f(g) = (g')$ is equivalent to $g^{-1}g' \in H$, the quotient space $\pi^{-1}(H')/H$ is homeomorphic to the compact space $\mathcal{D} \cap xK$, which completes the proof of the theorem.

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