Bull. Korean Math. Soc.
Vol. 17, No1, 1980

CONSTRUCTION OF APPROXIMATE SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY PARAMETRICES

BY JONGSIK KIM*)

The purpose of this paper is to construct distribution solutions of given elliptic or strongly hyperbolic partial differential equations, modulo C^∞ functions. For the construction of such solutions we shall use pseudodifferential operators and Fourier integral operators developed in [1], [2] and [4]. The solutions thus obtained are approximate ones. But investigations of such solutions clarify many properties of exact solutions such as propagations of singularities. We shall depend heavily on the techniques of constructing parametrices of linear partial differential operators developed by F. Treves in [7].

§1. Preliminaries.

Throughout the forthcoming we shall denote by Ω an open subset of R^n. A (linear partial) differential operator in Ω will be an operator of the form

$$P(X, D)=\sum_{|\alpha|\leq m} C_\alpha(X)D^\alpha \tag{1.1}$$

where the coefficients C_α are complex valued C^∞ functions in Ω. We have used the standard multi-index notations:

$\alpha=(\alpha_1, \cdots, \alpha_n)$, $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, $D_i=-i(\partial/\partial x_i)$ $|\alpha|=\alpha_1+\cdots+\alpha_n$.

We assume that the order of $P(X, D)$ is m and shall denote by $P_m(x, D)$ the principal part of $P(x, D)$.

Let u and v be distributions in Ω. If $u-v\in C^\infty(\Omega)$, we write $u\sim v$ and shall say that u is *equivalent to v modulo a C^∞ function.*

We shall rapidly recall the definitions of pseudodifferential operators and Fourier integral operators with some related concepts. For details we refer to [2], [4].

DEFITION 1.1. We denote by $S^m(\Omega, \Omega)$ the linear subspace of C^∞ functions in $\Omega\times\Omega\times R_n$, which has the following property; *to every compact subset K of $\Omega\times\Omega$ and every triplet of n-tuples p, q, r, there is a constant $C_{p,q,r}(K)>0$ such that*

$$|D_\xi^p D_x^q D_y^r a(x, y, \xi)|\leq C_{p,q,r}(K)(1+|\xi|)^{m-|p|}. \tag{1.2}$$

*) This research is partially supported by the Ministry of Education.

Elements of $S^m(\Omega,\Omega)$ are called *symbols of order m.*

DEFINITION 1.2. Let $a(x,y,\xi)$ be a symbol in $S^m(\Omega,\Omega)$. The operator A from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$ defined by

$$Au(x)=(2\pi)^{-n}\iint e^{i(x-y)\cdot\xi}a(x,y,\xi)u(y)\,dyd\xi \tag{1.3}$$

for any $u\in\mathcal{E}'(\Omega)$ is called a *pseudodifferential operator.* $a(x,y,\xi)$ is called a *symbol* of A

DEFINITION 1.3. An operator from $\mathcal{D}'(\Omega)$ (or $\mathcal{E}'(\Omega)$) to $\mathcal{D}'(\Omega)$ is called a *regularizing operator* if its image belongs to $C^\infty(\Omega)$.

We recall that every pseudodifferential operator can be extended to an operator from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Omega)$ modulo a regularizing operator. That is, for any pseudodifferential operator $A:\mathcal{E}'(\Omega)\to\mathcal{D}'(\Omega)$, there is a pseudodifferential operator $B:\mathcal{E}'(\Omega)\to\mathcal{D}'(\Omega)$ such that $B-A$ is regularizing and B can be extended to a continuous linear operator from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$.

DEFINITION 1.4. Let d be a real number. A function $\phi\in S^d(\Omega,\Omega)$ is said to be a *phase function* if it is real and if there is a number $C>0$ such that, for $|\xi|$ large,

$$|\partial_{x,y}\phi|^{-2}\partial_{x,\xi}\phi \text{ and } |\partial_{y,\xi}\phi|^{-2}\partial_{y,\xi}\phi \text{ belong to } S^{-c}(\Omega,\Omega;R^{2n}), \tag{1.4}$$

where $\partial_{x,\xi}\phi=(\partial_x\phi, |\xi|\partial_\xi\phi)$.

DEFINITION 1.5. Let $\phi\in S^d(\Omega,\Omega)$ be a phase function and $a\in S^m(\Omega,\Omega)$. Then the operator from $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$ defined by

$$Fu(x)=(2\pi)^{-2}\iint e^{i\phi(x,y,\xi)}a(x,y,\xi)u(y)\,dyd\xi \tag{1.5}$$

for any $u\in\mathcal{E}'(\Omega)$ is called a *Fourier integral operator.*

§2. Elliptic linear partial differential equations

In this section we shall construct parametrices of elliptic differential operators and solve, modulo C^∞ functions, elliptic differential equations.

DEFINITION 2.1. The differential operator $P(x,D)$ is said to be *elliptic* in Ω if, for every $x\in\Omega$, $P(x,\xi)=0$, $\xi\in R_n$ implies $\xi=0$.

THEOREM 2.1. *Let $P(x,D)$ be an elliptic differential operator. Then there exists a pseudodifferential operator K, called a parametrix of $P(x,D)$, such that $PK\sim I$ modulo a regularizing operator. The symbol of K is $\sum_{j=0}^\infty b_j$, where*

$$b_0=1/p(x,\xi), \tag{2.1}$$

$$b_j = -(1/P(x,\xi))\sum_{j'=0}^{j-1}\sum_{|p|=j-j'}(1/p!)\partial_\xi^p pD_x^q b_{j'} \quad (j>0) \tag{2.2}$$

Proof. It is enough to find a pseudodifferential operator K with symbol b such that

$$\Sigma(1/p!)\partial_\xi^p pD_x^q b = 1 \tag{2.3}$$

since the symbol ofPK is given by $\sum_p (1/p!)\partial_\xi^p pD_x^q b$ (*cf.* [5]).

Setting $b=\sum_{j=0}^\infty b_j$, we can determine the b_j, successively by the equations (2.3) to get (2.1) and (2.2). Since the homogeneous degree of b_j in ξ goes to $-\infty$ as $j\to\infty$, $\sum_j b_j$ defines a symbol in $S^{-m}(\Omega,\Omega)$ and we get $PK\sim I$ modulo a regularizing operator. (Q. E. D.)

THEOREM 2.2. *Let $P(x,D)$ be an elliptic operator and K be a parametrix of P as in the Theorem 2.1. Then the distribution solution u of*

$$P(x,D)u=v \ \ in \ \Omega \tag{2.4}$$

for any distribution v is given by $u\sim Kv$ in Ω modulo C^∞ function.

Proof. We may assume that P and K are defined on $\mathcal{D}'(\Omega)$. Since $PK\sim I$, we have $P(x,D)Kv\sim v$ modulo C^∞ functions. If $P(x,D)u=v$, then $P(x,D)(Kv-u)$ is a C^∞ function. Since $P(x,D)$ is hypoelliptic, $Kv-u$ is a C^∞ function. Thus $u\sim Kv$ modulo a C^∞ function. (Q. E. D.)

§ 3. Strongly hyperbolic linear partial differential equations

Our purpose in this section is to solve locally, modulo C^∞ functions, strongly hyperbolic equations with some constraints on the data. Thus we think of the differential operator

$$P(x,t,D_x,\partial_t) = \partial_t^m + \Sigma_{j=1}^m P_j(x,t,D_x)\partial_t^{m-j} \tag{3.1}$$

where $x=(x^1,\cdots,x^n)$ belongs to R^n, t to an open interval $(-T,T)$ and P_j is a differential operator of order j in the x-variables. We always assume that P has C^∞ coefficients in x and t variables.

DEFINITION 3.1. We say that the operator $P(x,t,D_x,\partial_t)$ is *strongly hyperbolic* in $\Omega\times(-T,T)$ if, for every point (x_0,t_0) of this open set and $\xi\in R_n\setminus\{0\}$, the constant coefficient polynomial $p_m(x_0,t_0,\xi,\tau)$, the principal part of $p(x,t,\xi,\tau)$, has m distinct, purely imaginary roots in τ.

We seek a distribution $u(x,t)$ in $\Omega\times(-T,T)$ satisfying, modulo C^∞ functions, the following Cauchy problem:

$$P(x,t,D_x,\partial_t)u=f \ \ in \ \Omega\times(-T,T) \tag{3.2}$$

$$\partial_t^j u|_{t=0}=u_j \text{ in } \Omega, \quad u=0, \ldots, m-1, \tag{3.3}$$

where f is an element in $C^\infty(\Omega\times(-T, T))$ with the compact x-projection of the support of f, independently from t, u_j is an element in $\mathcal{E}'(\Omega)$ for each j and $P(x, t, D_x, \partial_t)$ is a strongly hyperbolic differential operator.

By the well known Duhamel's principle, the solution of (3.2) and (3.3), which is unique, is given by

$$u(x, t)=\sum_{j=0}^{m-1} E_j(t)u_j(t)+\int_0^t E_{m-1}(t, t')f(x, t')dt' \tag{3.4}$$

where, for each $j=0, 1, \ldots, m-1$, $E_j(t)=E_j(t, 0)$ and $E_j(t, t')$ is the solution of

$$P(x, t, D_x, \partial_t)E_j(t, t')=0, \quad -T<t<T \tag{3.5}$$

$$\partial_t{}^k E_j(t, t')|_{t=t'}=\begin{cases}0 \text{ if } k\neq j, \\ 1 \text{ if } k=j.\end{cases} \quad (k=0, 1, \ldots m-1) \tag{3.6}$$

In (3.5) and (3.6), $E_j(t, t')$ is a smooth function of t and t' in $(-T, T)$ with values in the space of linear operotors on $\mathcal{E}'(\Omega)$.

We shall, in the following, represent $E_j(t)$ for $j=0, \ldots, m-1$ and $t\in[-T_0, T_0]$ for some T_0, $0<T_0<T$, by a sum of Fourier integral operators $F_{jk}(t)$ modulo regularizing operators. Once this is done, $E_j(t)$, hence $E_j(t, t')$, maps $C_c^\infty(\Omega)$ into $C^\infty(\Omega)$ as a general Fourier integral operator does. Thus the last term in (3.4) becomes a C^∞ function, giving rise to a required solution, mudulo C^∞ function, of (3.2) and (3.3); namely,

$$u(x, t)\sim\sum_{j=0}^{m-1}\left(\sum_{k=1}^{m} F_{jk}(t)\right)u_j(t) \tag{3.7}$$

Since $P_m(x, t, \xi, \tau)$ has m distinct, purely imaginary roots whatever (x, t) in $\Omega\times(-T, T)$ and $\xi\in R_n\setminus\{0\}$, we may denote them by $i\lambda_k(x, t, \xi)$, $k=1, \ldots, m$, with the agreement that $\lambda_1<\lambda_2<\cdots<\lambda_m$.

THEOREM 3.1. *Let ϕ_k $(k=1, 2, \ldots, m)$ be a solution of*

$$\partial_t\phi_k=\lambda_k(x, t, \partial_x\phi_k), \tag{3.8}$$

$$\phi_k|_{t=0}=x\cdot\xi \tag{3.9}$$

where $\lambda\in C^\infty(\Omega\times(-T, T)\times R_n\setminus\{0\})$ is a real valued and positive homogeneous of degree one with respect to ξ. Then for some T_0, $0<T_0<T$, ϕ_k is a phase function for any $t\in(-T_0, T_0)$.

Proof. The solution ϕ_k is real valued and, because of the uniqueness of the solution and of the fact that the initial datum $x\cdot\xi$ is homogeneous of degree one in ξ, ϕ_k is positive-homogeneous of degree one in ξ. Thus

$\phi_k \in S^1\ (\Omega, \Omega)$. Now if we set, for large $|\xi|$,

$$w = w(x, y, t, \xi) = \phi_k(x, t, \xi) - y\cdot\xi,$$

we have $\partial_x w(x, y, 0, \xi) = \partial_x \phi_k(x, 0, \xi) = \xi$.

Hence we may find $T_0 > 0$ such that

$$|\partial_x w| \geq |\xi|/2 \text{ for all } x \in \Omega, \quad |t| < T_0.$$

On the other hand, it is clear that $\partial_y w = \xi$. Thus both $|\partial_x w|$ and $|\partial_y w|$ are elliptic for $|t| < T_0$ in the whole $\Omega \times \Omega$ and so are $|\partial_{x,\xi}\phi_k|^2$ and $|\partial_{y,\xi}\phi_k|^2$. Therefore for $|t| < T_0$, (1.4) holds for $c = d$ and thus ϕ_k is a phase function. (Q. E. D.)

Now in our case, since the given data of the Cauchy problem are distributions with compact support, we may assume that (3.8) has a solution in Ω since it is always locally solvable.

Thus for each linear factor $(\partial_t - i\lambda_k)$ of P we constructed a phase function $\phi_k \in S^1(\Omega, \Omega)$. We shall determine symbols $a_{jk}(x, t, \xi)$ such that

$$\begin{aligned} E_j(t)u(x) &\sim \sum_{k=1}^{m} (2\pi)^{-n} \int e^{i\phi_k} a_{jk}(x, t, \xi)\hat{u}(\xi)\,d\xi \\ &= \sum_{k=1}^{m} (2\pi)^{-n} \iint e^{i(xy-\phi_k)} a_{jk}(x, t, \xi) u(y)\,dy d\xi, \quad (|t| < T_0) \end{aligned} \tag{3.10}$$

In view of (3.5) it suffices to determine a_{jk} to satisfy

$$\begin{aligned} 0 &= P(x, t, D_x, \partial_t) E_j(t) u(x) \sim \\ &\sum_{k=1}^{m} (2\pi)^{-n} \int e^{i\phi_k} P(x, t, D_x + \partial_x\phi_k, \partial_t + i\partial_t\phi_k) a_{jk}(x, t, \xi)\hat{u}(\xi)\,d\xi \end{aligned} \tag{3.11}$$

Since $e^{i\phi_k}(k=1, 2, \ldots, m)$ are linearly independent, we may require

$$P(x, t, D_x + \partial_x\phi_k, \partial_t + i\partial_t\phi_k) a_{jk}(x, t, \xi) = 0 \tag{3.12}$$

Now let us set

$$a_{jk}(x, t, \xi) = \sum_{l=0}^{\infty} a_{jkl}\ (x, t, \xi), \tag{3.13}$$

where a_{jkl} is homogeneous with respect to ξ whose homogeneous degree decreases to $-\infty$ as $l \to \infty$. Since $P_m(x, t, \partial_x\phi_k, i\partial_t\phi_k) = 0$, from (3.12)

$$\begin{aligned} &(\partial_t P_m)(x, t, \partial_x\phi_k, i\partial_t\phi_k)\partial_t a_{jk0} \\ &\quad + \sum_{\nu=1}^{n} (\partial_{\xi_\nu} P_m)(x, t, \partial_x\phi_k, i\partial_t\phi_k) D_{x_\nu} a_{jk0} + \tilde{c}(\phi_k; x, t, \xi) a_{jk0} = 0, \end{aligned} \tag{3.14}$$

where $\tilde{c}(\phi_k; x, t, \xi)$ is the coefficient of order $m-1$ in ξ.

On the other hand, from (3.8),

$$(\partial_\tau P_m)(x, t, \partial_x\phi_k, i\partial_t\phi_k) = i^{m-1}\sum_{k'\neq k}\{\lambda_k(x, t, \partial_x\phi_k) - \lambda_{k'}(x, t, \partial_x\phi_k)\} \quad (3.15)$$

and

$$(\partial_{\xi_\nu} P_m)(x, t, \partial_x\phi_k, i\partial_t\phi_k) = i(\partial_t P_m)(x, t, \partial_x\phi_k, i\partial_t\phi_k) C_{k,\nu}(x, t, \xi), \quad (3.16)$$

where

$$C_{k,\nu}(x, t, \xi) = (\partial_{\xi_\nu}\lambda_k)(x, t, \partial_x\phi_k). \quad (3.17)$$

We write

$$C_{k,0}(x, t, \xi) = \tilde{C}(\phi_k; x, t, \xi) / (\partial_\tau P_m)(x, t, \partial_x\phi_k, i\partial_t\phi_k). \quad (3.18)$$

Then the equation (3.14) reads, now:

$$\partial_t a_{jk0} - \sum_{\nu=1}^{n} C_{k,\nu}(x, t, \xi)(\partial_{x_\nu} a_{jk0} + C_{k,0}(x, t, \xi) a_{jk0} = 0. \quad (3.19)$$

The a_{jkl} ($l>0$) are determined successively by the following equation;

$$\partial_t a_{jkl} - \sum_{\nu=1}^{n} C_{k,\nu}(x, t, \xi)\partial_{x_\nu} a_{jkl} + C_{k,0}(x, t, \xi) a_{jkl}$$
$$= \sum_{k'=1}^{m} Q_{k,k'}(x, t, \xi, \partial_x, \partial_t) a_{jk(l-k')}, \quad (3.20)$$

where $Q_{k,k'}$ are differential operators whose expressions can be computed from (3.14).

To determine a_{jk} concretely we require appropriate conditions at time $t=0$. In virtue of (3.6), a_{jkl} has to satisfy

$$\partial_t^{j'} E_j(t) u(x) \sim \sum_{k=1}^{m} (2\pi)^{-n} \int e^{i\phi_k} (\partial_t + i\partial_t\phi_k)^{j'} a_{jk}(x, t, \xi) \hat{u} d\xi. \quad (3.21)$$

Therefore it suffices to find a_{jkl} such that

$$\sum_{k=1}^{m} (\partial_t + i\partial_t\phi_k)^{j'} a_{jk}|_{t=0} = \begin{cases} 1 \text{ if } j'=j, \\ 0 \text{ if } j'\neq j. \end{cases} \quad (3.22)$$

Substituting $a_{jk} = \sum_{l=0}^{\infty} a_{jkl}$, since $\partial_t\phi_k = \lambda_k(x, 0, \xi)$ when $t=0$, from (3.22) we get

$$\sum_{k=1}^{m} (i\lambda_k) j' a_{jk0}|_{t=0} = \begin{cases} 1 \text{ if } j'=j, \\ 0 \text{ if } j'\neq j. \end{cases} \quad (3.23)$$

Let $V(\tau_1, \cdots, \tau_m)$ be the Vandermonde determinant with respect to $\tau_1, \cdots, \tau_m$, Then

$$a_{jk0} = V_{jk}(i\lambda_1, \cdots, \widehat{\tau_k}, \cdots, i\lambda_m) / V(i\lambda_1, \cdots, i\lambda_m), \quad (3.24)$$

when $t=0$.

Here we denoted by $V_{jk}(\tau_1, \cdots, \widehat{\tau_k}, \cdots, \tau_m)$ the minor of the term $\tau_k{}^j$. a_{jko} is of homogeneous degree $-j$ in ξ and thus belongs to $S^{-j}(\Omega, \Omega)$.

The a_{jkl} $(l<0)$ are determined successively by the equations

$$\sum_{k=1}^{m}(i\lambda_k)^{j'}a_{jkl}=\sum_{j=1}^{m}\ \sum_{j''=1}^{j'}R^{j'}_{j''k}(x, \xi, \partial_t)\,a_{jk(l-j'')} \tag{3.25}$$

where for each $j''=1, 2, \ldots, j'$ $R^{j'}_{j''k}(x, \xi, \partial_t)$ is a polynomial of degree $\leq j''$ with respect to ∂_t, which can be computed from (3.22). a_{jkl} is of homogeneous degree $-j-l$ and thus belongs to $S^{-j-l}(\Omega, \Omega)$ and $\sum_{l=0}^{\infty}a_{jkl}$ defines a symbol belonging to $S^{-j}(\Omega, \Omega)$.

Summarizing the above argument, we get

THEOREM 3.2. *Let a_{jk} be symbols determined by* (3.19)-(3.24) *and* (3.20)-(3.25). *Let $a_{jk}=\sum_{l=0}^{\infty}a_{jkl}$ and let ϕ_k be as in the Theorem* 3.1. *If F_{jk} is the Fourier integral operator defined by*

$$F_{jk}u(x)=(2\pi)^{-n}\iint e^{i(x\cdot y-\phi_k)}a_{jk}(x,\ t, \xi)u(y)\,dy\,d\xi,$$

then for suitable T_0 $(0<T_0<T)$ determined as in the Theorem 3.1, *the solution $u(x, t)$ of* (3.2)-(3.3) *is given by*

$$u(x, t)\sim\sum_{j=0}^{m-1}\Big(\sum_{k=1}^{m}F_{jk}(t)\Big)u_j(x)$$

for $|t|<T_0$.

References

1. J. Duistermaat, *Fourier Integral Operators,* New York Univ., 1973.
2. J. Duistermaat and L. Hörmander, *Fourier Integral Operators II,* Acta Math. **128** (1972) 183-269.
3. L. Gàrding, *Linear Hyperbolic Partial Differential Equations with Constant Coefficients,* Acta Math. **85**(1950), 1-62.
4. L. Hörmander, *FourierIntegral Operators I,* Acta Math., **127**(1971), 79-183.
5. J. Kohn and L. Nirenberg, *An algebra of pseudodifferential operators,* Comm. Pure Appl. Math. **18**(1965), 269-305.
6. Lecture Notes in Math., *Fourier Integral Operators and Partial Differential Equations,* Springer Verlag, 1975.
7. F. Treves, *Pseudodifferential Operators and Fourier Integral Operators* Vol. I & II, will be published by Plenum Pub. Co.

Seoul National University