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ON A STABLY FREE MODULE

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1. Introduction

Let R be an integral domain of characteristic 0 with quotient field L and Π a finite group such that no rational prime dividing the order of Π is a unit in R. Let T be a free abelian group or monoid of finite rank.

We prove that *for any finitely generated projective* $R[T][\Pi]$*-module* P, $L\otimes_R P$ *is* $L[T][\Pi]$*-stably free.*

This result is a generalization of a theorem of Sharma [4, p. 303].

2. Notations and Definitions

Each ring considered in this article will be assumed to be an associative ring with 1 and all ring homomorphisms as well as all modules are unitary.

For a finite group Π by $\mathrm{ord}(\Pi)$ we mean the order of Π.

For a ring A, $\mathfrak{P}(A)$ denotes the class of all finitely generated projective (left) A-modules.

Let $P\in\mathfrak{P}(A)$. Then P is said to be *stably free over* A, if there exists a finitely generated free A-module F such that $P\oplus F$ is free over A. It is known that a finitely generated projective A-module P is stably free if and only if $P\in\mathbf{Z}[A]$ in $K_0(A)$ (Lam [2, p. 40]).

Let R be a commutative ring and $\mathfrak{p}\in\mathrm{Spec}(R)$, the prime spectra of R. Then the minimal number of generators of $P_{\mathfrak{p}}=P\underset{R}{\otimes}R_{\mathfrak{p}}$ as a $P_{\mathfrak{p}}$-module is said to be *the rank of* $P_{\mathfrak{p}}$ *over* $R_{\mathfrak{p}}$. We denote it by $\mathrm{rk}_{\mathfrak{p}}P$.

We shall say that P *has constant rank* r and write $\mathrm{rk}_R P=r$ if for every $\mathfrak{p}\in\mathrm{Spec}(A)$ $\mathrm{rk}_{\mathfrak{p}}P=r$.

Let A be a local ring (not necessarily commutative) and $P\in\mathfrak{P}(A)$. Then the minimal number of generators of P is said to be the *rank* of P over A and it is alsodenoted by $\mathrm{rk}_A P$.

Let Π be a finite group, A a ring and $\mathcal{C}$ be the class of all cyclic subgroups of Π. Then $K_0(A\Pi)^{\mathcal{C}}$ denotes

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$$\bigcap_{\substack{\Pi' \subset \Pi \\ \Pi' \in \mathfrak{C}}} \ker [K_0(A\Pi) \xrightarrow{i^*} K_0(A\Pi')].$$

See Swan [5, p. 23].

3. Preliminary Results

PROPOSITION 3.1. *Let R be an integral domain of characteristic 0, Π a finite group of order n and $P \in \mathfrak{P}(R\Pi)$, If no rational prime dividing n is a unit in R, then n divides $\mathrm{rk}_R P$.*

Proof : Let p be a rational prime dividing n and Π_p be a Sylow p-subgroup of Π. Then $P \in \mathfrak{P}(R(\Pi_p))$. Hence if the theorem is true for p-groups it is true for finite groups. Thus we may assume that Π is a p-group.

Since p is not a unit in R, there is a maximal ideal $\mathfrak{M}$ in R with $p \in \mathfrak{M}$.

$$P/\mathfrak{M}P = (R/\mathfrak{M}) \otimes_R P \in \mathfrak{P}((R/\mathfrak{M})\Pi)$$

implies that $P/\mathfrak{M}P$ is free over $(R/\mathfrak{M})\Pi$ (Swan [6, p. 58]). Thus we have

$$\dim_{R/\mathfrak{M}}(P/\mathfrak{M}P) = n \cdot \mathrm{rk}_{R/\mathfrak{M}\Pi}(P/\mathfrak{M}P).$$

But $\mathrm{rk}_R P = \mathrm{rk}_{R/\mathfrak{M}}(R/\mathfrak{M} \otimes_R P) = \dim_{R/\mathfrak{M}}((R/\mathfrak{M}) \otimes^R P) = \dim_{R/\mathfrak{M}}(P/\mathfrak{M}P)$

This completes the proof.

COROLLARY 3.2. *Let R be an integral domain of characteristic 0 and Π an finite abelian group such that no rational prime dividing the order of Π is a unit in R. Then $P \in \mathfrak{P}(R\Pi)$ has constant rank.*

Proof : It is enough to show that $R\Pi$ has no idempotent other than 0 and 1. If e is an idempotent in $R\Pi$, then we have the following direct sum

$$R\Pi = R\Pi e \oplus R\Pi(1-e).$$

Let $P = R\Pi e$, $Q = R\Pi(1-e)$. Then the order of Π divides $\mathrm{rk}_R P$ and $\mathrm{rk}_R Q$ by Proposition 3.1. But

$$\mathrm{ord}(\Pi) = \mathrm{rk}_R(R\Pi) = \mathrm{rk}_R P + \mathrm{rk}_R Q.$$

Therefore, $\mathrm{rk}_R P = 0$ or $\mathrm{rk}_R Q = 0$. Hence $e = 0$ or 1.

PROPOSITION 3.3. *Let $f: R \to S$ be a homomorphism of commutative rings. If no idempotent in R is in the kernel of f, then $H(f): H(R) \to H(S)$ is injective (Swan [4, p. 138]).*

Proof. Let g be an element of the kernel of $H(f)$. Then for every ideal $\mathfrak{q} \in \mathrm{Spec}(S)$, $0 = H(f)(g)(\mathfrak{q}) = g(f^{-1}\mathfrak{q})$. Thus it is enough to show that for each $\mathfrak{p} \in \mathrm{Spec}(R)$, $g(\mathfrak{p}) = 0$.

Let n be a rational integer with $g(\mathfrak{p}) = n$. We need only to show that $n = 0$. Let $X = g^{-1}(n)$, then X is open and closed in $\mathrm{Spec}(R)$ and so there is an idempotent e in R such that $X = V(I)$, a Zariski closed set in $\mathrm{Spec}(R)$, where I is an ideal in R generated by $1-e$. By the given hypothesis we have $f(e)\neq 0$, thus $1-f(e)$ is a non-zero idempotent and so not a unit in S. Hence there is a maximal ideal $\mathfrak{M}$ in S with $1-f(e)\in\mathfrak{M}$. Thus $1-e\in f^{-1}(\mathfrak{M})$ and so $f^{-1}(\mathfrak{M})\in X$. Therefore $n=g(f^{-1}(\mathfrak{M}))=H(f)(g)(\mathfrak{M})$. But $H(f)(g)=0$. Hence $n=0$.

PROPOSITION 3.4. *Let A be a left regular ring and let T be a free abelian group or monoid. Then the canonical homomorphism $K_0(A)\to K_0(A[T])$ is an isomorphism* (Bass [1, p. 636]).

4. Main Theorems

THEOREM 4.1. *Let R be an integral domain of characteristic 0 with quotient field L and let S be a subring of L with $R\subset S\subset L$ and Π a finite group of order n such that no prime dividing n is a unit in R. Furthermore, we assume:*

(1) *For each cyclic subgroup Π' of Π every finitely generated projective $S\Pi'$-module with constant rank is stably free.*

(2) *$K_0(S\Pi)$ is torsion free.*

Then $P\in\mathfrak{P}(R\Pi)$ implies $S\otimes_R P$ is stably free over $S\Pi$.

Proof. Let $P\in\mathfrak{P}((R\Pi))$. Then $\mathrm{rk}_R P=rn$ for some positive rational integer r by Proposition 3.1. We claim that $[S\otimes_R P]=r[S\Pi]$ in $K_0(S\Pi)$. We know that $n^2K_0(S\Pi)^{\mathfrak{C}}=0$ (Swan[6. pp. 23, 25]). But, by the given hypothesis (2), $K_0(S\Pi)$ is torsion free. Hence $K_0(S\Pi)^{\mathfrak{C}}=0$. Therefore it is enough to show that $[S\underset{R}{\otimes}P]-r[S\Pi]\in K_0(S\Pi)^{\mathfrak{C}}$. For each cyclic subgroup Π' of Π, P has constant rank over $R\Pi'$ by Corollary 3.2. Let $R\Pi'\to S\Pi$ be the canonical ring homomorphism, $\mathfrak{q}\in\mathrm{Spec}(S\Pi')$ and let $\mathfrak{p}=f^{-1}(\mathfrak{q})$. Then we have the following canonical homomorphism

$$(R\Pi')_{\mathfrak{p}}\to(S\Pi')_{\mathfrak{q}}$$

which is induced by f.

Note the following canonical isomorphisms

$$\begin{aligned}(S\Pi')_{\mathfrak{q}}\underset{S\Pi'}{\otimes}(S\otimes_R P)&\cong(S\Pi')_{\mathfrak{q}}\underset{S\Pi'}{\otimes}(S\Pi'\underset{R\Pi'}{\otimes}P)\\&\cong(S\Pi')_{\mathfrak{q}}\underset{(R\Pi')_{\mathfrak{p}}}{\otimes}((R\Pi')_{\mathfrak{p}}\underset{R\Pi'}{\otimes}P)\cong(S\Pi')_{\mathfrak{q}}\underset{(R\Pi')_{\mathfrak{p}}}{\otimes}F\end{aligned}$$

where F is a free $(R\Pi')_{\mathfrak{p}}$-module on $\mathrm{rk}_{R\Pi'}\mathfrak{p}$ generators. This shows that $S\underset{R}{\otimes}P$ has constant rank over $S\Pi'$. Therefore $[S\underset{R}{\otimes}P]=m\cdot[S\Pi']$ in $K_0(S\Pi')$, for some positive rational integer m, by the hypothesis (1). We will prove below that $m=(\Pi:\Pi')\cdot r$ where $(\Pi:\Pi')$ is the index of Π' in Π. We have

$[S\otimes_R P]=m\cdot[S\Pi']=r\cdot(\Pi:\Pi')\cdot[S\Pi']=r\cdot[S\Pi]$ in $K_0(S\Pi')$.

Thus $[S\otimes_R P]-r[S\Pi]\in K_0(S\Pi)^e$. But $K_0(S\Pi)^e=0$. Hence $S\underset{R}{\otimes}P$ is stably free over $S\Pi$. It now remains to be shown that $m=r\cdot(\Pi:\Pi')$. Since $[S\underset{R}{\otimes}P]=m\cdot[S\Pi']$ in $K_0(S\Pi')$, we have $[L\underset{R}{\otimes}P]=m\cdot[L\Pi']$ in $K_0(L\Pi')$ and so $[L\underset{R}{\otimes}P]=m\cdot\mathrm{ord}(\Pi')\cdot[L]$ in $K_0(L)$. But $\mathrm{rk}_S P=r\cdot\mathrm{ord}(\Pi)=m\cdot\mathrm{ord}(\Pi')$, for $K_0(L)$ is torsion free. Therefore, we have $m=r\cdot(\Pi:\Pi')$.

THEOREM 4.2. *Let R be an integral domain of characteristic 0 with quotient field L. Let Π be a finite group such that no rational prime dividing the order of Π is a unit in R. Let T be afree abelian group or monoid of finite rank. Then for every $P\in\mathfrak{P}[R[T][\Pi]]$, $L\otimes_R P$ is stably free over $L[T][\Pi]$.*

Proof. Let m be the order of Π and $\mathbf{N}$ be the set of non-negative rational integers. Since the following map $R[\Pi][T]\to R[T][\Pi]$,

$$\sum_{v\in N^n}(\sum_{g\in\Pi}\tau_{vg}g)T^v\longrightarrow\sum_{g\in\Pi}(\sum_{V\in N^n}\tau_{vg}T^v)g$$

is an isomorphism of rings, we may identify $R[\Pi][T]$ and $R[T][\Pi]$. Since no rational prime dividing the order of Π is a unit in $R[T]$, we only need to show that $L[T]$ satisfies the two conditions of Theorem 4.1. Since $L\Pi$ is semisimple, it is left regular. Therefore we have the following isomorphism

$$K_0(L[T][\Pi])=K_0(L[\Pi][T])\cong K_0(L\Pi)$$

by Proposition 3.4. Furthermore, $K_0(L\Pi)$ is torsion free. Now it is enough to show that for each abelian subgroup Π' of Π every finitely generated projective $L[T][\Pi']$-module with constant rank is stably free.

From now on we may assume Π is abelian. Let $f: L\Pi\to L\Pi[T]$ be the canonical injection. Then the following diagram of canonical homomorphisms

$$\begin{array}{ccc} K_0(L\Pi[T]) & \xrightarrow[\cong]{K_0(f)} & K_0(L\Pi) \\ \downarrow r & & \downarrow r \\ H(L\Pi[T]) & \underset{H(f)}{\longleftarrow} & H(L\Pi) \end{array}$$

is commutative (Swan [5, p. 138]). But $H(f)$ is injective by Proposition 3.3. Furthermore, $r: K_0(L\Pi)\to H(L\Pi)$ is also injective. For an element x of the kernel of the map r, there are $P, F\in\mathfrak{P}(L\Pi)$ such that F is free and $x=[P]-[F]$ in $K_0(L\Pi)$. Thus for each $\mathfrak{p}\in\mathrm{Spec}(L\Pi)$, $r_P(\mathfrak{p})=\mathrm{rk}_{L\Pi}(F)$. Hence P has constant rank. Since $L\Pi$ is semisimple it is semilocal and P

is free. This implies that $x=[P]-[F]=0$, and so the map r is injective. Now it is clear that

$$r: K_0(L\Pi[T]) \longrightarrow H(L\Pi[T])$$

is also injective. Let $P \in \mathfrak{P}(L\Pi[T])$ with constant rank (Bourbaki [2, § 5.3, Prop. 5]). Let F be a free $L\Pi[T]$-module with $\mathrm{rk}_{L\Pi[T]}F=\mathrm{rk}_{L\Pi[T]}P$. Then $r_P=r_F$. But r is injective. Hence $[P]=[F]$ in $K_0(L\Pi[T])$.

This completes the proof.

References

1. Bass, H., *Algebraic K-Theory*, W. A. Benjamin, New York, 1968.
2. Bourbaki, N., *Algebre Commutative*, Herman, Paris, 1969.
3. Lam, T. Y., *Serre's Conjecture*, Springer Verlag, Berlin, 1978.
4. Sharma, P. K., *Projective Modules oves Group Rings*, J. Algebra **19** (1971), 303-314.
5. Swan, R. G., *Algebraic K-Theory*, Springer Verlag, Berlin, 1968.
6. Swan, R. G. *K-Theory of Finite Groups and Orders*, Springer Verlag, Berlin, 1970.

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