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ON GENERAL CONTRACTIVE TYPE CONDITIONS

BY SEHIE PARK*

1. Introduction

The well-known Banach contraction principle says that the Picard iteration of a point under a contraction of a complete metric space converges to its unique fixed point. A number of generalizations of this principle have appeared. A comparative study of these has been made by Rhoades [30].

However, recent works of Meir and Keeler [21], Daneš [7], Husain and Sehgal [13], [14], Hegedüs [11], Hegedüs and Szilágyi [12], and Kasahara [19] have extended the principle to wider classes of maps than those covered in [30]. These authors have defined contractive type conditions of the form

$$d(fx, fy) < \operatorname{diam} \ (O(x) \cup O(y))$$

where f is a selfmap of a metric space (X, d), $x, y \in X$, $x \neq y$, $\operatorname{diam} O(x) < \infty$, $\operatorname{diam} O(y) < \infty$, and $O(x) = \{f^n x \mid n \in \omega\}$, where $\omega = \{0, 1, 2, \ldots\}$, and have obtained interesting fixed point theorems.

In the present paper, we shall update Rhodes' comparative study of various results to the Banach contraction principle [30] and obtain some common extension of recent results along similar lines.

Section 2 deals with definitions and basic properties of generalized contractive type conditions. In Section 3, we extend some known fixed point theorems, and, in Section 4, we give an extension of a result in Section 3 to common fixed points of commuting selfmaps. In the final section, we indicate some open questions.

2. General contractive type conditions

Let f be a selfmap of a metric space (X, d). Given $x \in X$, let $O(x) = \{f^n x \mid n \in \omega\}$ and $\overline{O(x)}$ be its closure. A point $x \in X$ is said to be regular for f if $\operatorname{diam} O(x) < \infty$ [19]. Given $x, y \in X$, let

$$m(x, y) = \max \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\},$$

and

$$\delta(x, y) = \operatorname{diam} \{O(x) \cup O(y)\}$$

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whenever x and y are regular.

We list contractive type conditions to be considered, some of which are known and some of which are new.

(A) For any $x, y \in X$, $x \neq y$,

(Ad) $d(fx,fy) < d(x,y)$ (Edelstein [9]).

(Am) $d(fx,fy) < m(x,y)$ (Rhoades [30]).

(Aδ) if x and y are regular,

$$d(fx,fy) < \delta(x,y).$$

(B) Given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x, y \in X$,

(Bd) $\varepsilon \leq d(x,y) < \varepsilon + \delta$ implies $d(fx,fy) < \varepsilon$ (Meir-Keeler [21]).

(Bm) $\varepsilon \leq m(x,y) < \varepsilon + \delta$ implies $d(fx,fy) < \varepsilon$.

(Bδ) $\varepsilon \leq \delta(x,y) < \varepsilon + \delta$ implies $d(fx,fy) < \varepsilon$.

(C) Given $\varepsilon > 0$, there exist $\varepsilon_0 < \varepsilon$ and $\delta_0 > 0$ such that for any $x, y \in X$,

(Cd) $\varepsilon \leq d(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \leq \varepsilon_0$.

(Cm) $\varepsilon \leq m(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \leq \varepsilon_0$.

(Cδ) $\varepsilon \leq \delta(x,y) < \varepsilon + \delta_0$ implies $d(fx,fy) \leq \varepsilon_0$ (Hegedüs-Szilagyi [12]).

(D) There exists a nondecreasing right continuous function ϕ: $[0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ for $t > 0$ and, for any $x, y \in X$,

(Dd) $d(fx,fy) \leq \phi(d(x,y))$ (Browder [3]).

(Dm) $d(fx,fy) \leq \phi(m(x,y))$ (Danes [7]).

(Dδ) $d(fx,fy) \leq \phi(\delta(x,y))$ if x, y are regular (Kasahara [19]).

(E) There exists $\alpha \in [0, 1)$ such that for any $x, y \in X$,

(Ed) $d(fx, fy) \leq \alpha\, d(x,y)$ (Banach).

(Em) $d(fx, fy) \leq \alpha\, m(x,y)$ (Ciric [5], Massa [20]).

(Eδ) $d(fx, fy) \leq \alpha\, \delta(x,y)$ if x, y are regular (Hegedüs [11]).

Note that some variants of the conditions (Dm) and (Cm) appear in Husain-Sehgal [13], [14], Kasahara [18], and Park [25], [26].

If all points in X are regular for f, then we have the following diagram of implications:

$$\begin{array}{ccccccccc}
(\mathrm{Ad}) & \Longleftarrow & (\mathrm{Bd}) & \Longleftarrow & (\mathrm{Cd}) & \Longleftarrow & (\mathrm{Dd}) & \Longleftarrow & (\mathrm{Ed}) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
(\mathrm{Am}) & \Longleftarrow & (\mathrm{Bm}) & \Longleftarrow & (\mathrm{Cm}) & \Longleftarrow & (\mathrm{Dm}) & \Longleftarrow & (\mathrm{Em}) \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
(\mathrm{A}\delta) & \Longleftarrow & (\mathrm{B}\delta) & \Longleftarrow & (\mathrm{C}\delta) & \Longleftarrow & (\mathrm{D}\delta) & \Longleftarrow & (\mathrm{E}\delta)
\end{array}$$

It is interesting to note that the chronological order of the conditions is (Ed), (Ad), (Dd), (Bd), (Em), (Dm), (Am), (Eδ), (Dδ), (Cδ), and the others are newly given.

We prove only the nontrivial implications $(\mathrm{C}\delta) \Longleftarrow (\mathrm{D}\delta)$, $(\mathrm{Cm}) \Longleftarrow (\mathrm{Dm})$, and $(\mathrm{Cd}) \Longleftarrow (\mathrm{Dd})$.

$(\mathrm{C}\delta) \Longleftarrow (\mathrm{D}\delta)$. Let $\varepsilon>0$. Since $\phi(\varepsilon)<\varepsilon$, we can choose an ε_0 satisfying $\phi(\varepsilon)<\varepsilon_0<\varepsilon$. Since ϕ is nondecreasing and right continuous, there exists a $\delta_0>0$ such that $t\in[\varepsilon, \varepsilon+\delta_0]$ implies $\phi(t)\leq\varepsilon_0$. Suppose $\varepsilon\leq\delta(x, y)<\varepsilon+\delta_0$. Then we have $d(fx, fy)\leq\phi(\delta(x, y))\leq\varepsilon_0$.

Similar proofs can be applied for $(\mathrm{Cm}) \Longleftarrow (\mathrm{Dm})$ and $(\mathrm{Cd}) \Longleftarrow (\mathrm{Dd})$.

Counterexamples. Showing $(\mathrm{C}\delta) \not\Longrightarrow (\mathrm{D}\delta)$ is given by Hegedüs-Szilágyi [12], $(\mathrm{E}\delta) \not\Longrightarrow (\mathrm{Em})$ by Hegedüs [11], $(\mathrm{D}\delta) \not\Longrightarrow (\mathrm{E}\delta)$ by Kasahara [19], $(\mathrm{Dd}) \not\Longrightarrow (\mathrm{Ed})$ and $(\mathrm{Em}) \not\Longrightarrow (\mathrm{Ed})$ by many authors, and $(\mathrm{Bd}) \not\Longrightarrow (\mathrm{Cd})$ by Meir-Keeler [21].

Meir-Keeler [21] noted that if X is compact then $(\mathrm{Ad}) \Longleftrightarrow (\mathrm{Bd})$. We can show that $(\mathrm{Am}) \Longleftrightarrow (\mathrm{Bm})$ and $(\mathrm{A}\delta) \Longleftrightarrow (\mathrm{B}\delta)$ whenever X is compact and f is continuous.

$(\mathrm{A}\delta) \Longrightarrow (\mathrm{B}\delta)$. We consider $\inf_{\varepsilon\leq\delta(x,y)} [\delta(x, y)-d(fx, fy)]=\delta(\varepsilon)$, say. Since for each $i, j\in\omega$, $d(f^ix, f^jy)$ is continuous at $(x, y)\in X\times X$, we know that $\delta(x, y)=\sup_{i,j} d(f^ix, f^jy)$ is lower semicontinuous ([8], p. 85). Therefore, $\delta(x, y)-d(fx, fy)$ must attain its minimum at some point (a, b) in the compact space $X\times X$ ([8], p. 227), i. e., $\delta(\varepsilon)=\delta(a, b)-d(fa, fb)$.

If $a=b$, then $\delta(\varepsilon)=\delta(a, b)\geq\varepsilon$. If $a\neq b$, then, from $(\mathrm{A}\delta)$, $d(fa, fb)<\delta(a, b)$, and again $\delta(\varepsilon)>0$.

Now suppose $\varepsilon\leq\delta(x, y)<\varepsilon+\delta(\varepsilon)$. Then $\delta(x, y)-d(fx, fy)\geq\delta(\varepsilon)$ implies $d(fx, fy)+\delta(\varepsilon)\leq\delta(x, y)<\varepsilon+\delta(\varepsilon)$, which in turn implies that $d(fx, fy)<\varepsilon$.

Similarly, we can show that $(\mathrm{Am}) \Longrightarrow (\mathrm{Bm})$.

Furthermore, if f is a continuous compact map satisfying the condition (Am), then there exists an equivalent metric for X relative to which f satisfies (Ed) (See [15], [24], [31]). This will be extended to $(\mathrm{A}\delta)$ in the next section.

We list some simple observations for these general contractive type conditions:

(I) If f satisfies any contractive type condition in the list (A) to (E) and f has a fixed point, then it is unique.

(II) If there exists a positive integer k such that f^k satisfies any condition in the list (A) to (E) and f^k has a fixed point, then it is the unique fixed point of f.

(III) Any map satisfying (Ad) is uniformly continuous.

(IV) For any map f satisfying (Bd) or (Em) and for any $x\in X$, the Picard iteration $\{f^nx\}_{n\in\omega}$ is Cauchy (Meir-Keeler [21], Ćirić [5]). For any map f satisfying (Cδ) and for any regular $x\in X$, $\{f^n x\}$ is Cauchy (Hegedüs–Szilágyi [12]).

(V) A selfmap f of a metric space X is said to have diminishing orbital diameters if, for any $x\in X$, diam $O(x)$ satisfies the property that $0<$ diam $O(x)<\infty$ implies $\lim_n O(f^n x)<$diam $O(x)$ [2].

Meir–Keeler [21] have shown that condition (Bd) implies that f has diminishing orbital diameters. We will later note that the condition (Cδ) implies the same property (Lemma (i) to Theorem 2(Cδ)). Hence, by a result of Ng [23], a map f satisfying (Cδ) is non-periodic; that is, every periodic point is a fixed point.

(IV) Hegedüs [11] showed that if a selfmap f of a complete metric space X, all of whose points are regular, satisfies the condition (Eδ), then f is a contraction type map in the sense of Ćirić [6]; that is, f satisfies all conclusions of the Banach contraction principle. Hegedüs-Szilágyi [12] also obtained some equivalent conditions to each in (D) and (C) (Cf. [4], [7], [14], [19]).

A unified approach to fixed point theorems of maps satisfying the conditions (Dm) or (Ad) is given in [26].

3. Fixed Point Theorems

Edelstein [9] showed that if a selfmap f satisfies (Ad) and if, for some $u\in X$, $O(u)$ has a cluster point $p\in X$, then p is the unique fixed point of f and $f^n u\rightarrow p$. Rhoades [30] raised the question: if a continuous selfmap f satisfies (Am), and if, for some $u\in X$, $O(u)$ has a cluster point, does f possess a fixed point? Taylor [32] has constructed an example to show that the answer is in the negative. If, however, one adds the hypothesis that f is compact, then the answer is in the affirmative, as shown in [31]. In order to extend this result to a map satisfying (Aδ), we need the following.

Let $M(X)$ denote the set of all metrics on X that are topologically equivalent to d for a given metric space (X, d).

THEOREM (Meyers [22]). *Let f be a continuous selfmap of a metric space X with the following properties*:

(1) *f has a unique fixed point $p\in X$.*

(2) *For any $x\in X$, $f^n x\rightarrow p$.*

(3) *There exists an open neighborhood U of p with the property that given any open set V containing p there exists an integer n_0 such that $n>n_0$ implies $f^n U\subset V$. Then for any $\alpha\in(0,1)$ there exists a metric $\rho\in M(X)$ relative to which f satisfies the condition* (Ed).

We follow the method of Janos [15] and obtain

THEOREM 1. *Let f be a continuous compact selfmap of a metric space X satisfying* (Aδ). *Then f has a unique fixed point, and furthermore, for any $\alpha \in (0, 1)$ there exists a metric ρ in $M(X)$ relative to which f satisfies* (Ed) *with the Lipschitz constant α.*

Proof. There exists a compact subset Y of X containing fX. Then $fY \subset Y$ and, hence, $A = \bigcap_{n=1}^{\infty} f^n Y$ is a nonempty compact f-invariant subset of X which is mapped by f onto itself. We claim that A is a singleton consisting of the unique fixed point p of f. If not, then diam $A > 0$. Since A is compact, there exist $x, y \in A$ with $d(x, y) = \operatorname{diam} A$, and, since f maps A onto itself, there exist $x', y' \in A$ with $x = fx'$, $y = fy'$. Since f satisfies (Aδ) we have

$$\operatorname{diam} A = d(x, y) = d(fx', fy') < \delta(x', y') \leq \operatorname{diam} A,$$

a contradiction. Therefore condition (1) of Meyers' theorem holds. Condition (2) follows from the fact that $f^n X \subset Y$ for any n. For (3), take $U = X$ and observe that $f^{n+1}X \subset f^n X$, whose diameter diminishes to zero as $n \to \infty$. Thus $f^n X$ squeezes into any neighborhood of p and the proof is complete.

Now we prove a theorem for a map satisfying condition (Cδ).

THEOREM 2 (Cδ). *Let f be a selfmap of a metric space X. Suppose there exists a regular point $u \in X$ such that*

(1) *$O(u)$ has a regular cluster point $p \in X$, and*

(2) *the condition* (Cδ) *holds on $O(u) \cup O(p)$.*

Then f has a unique fixed point p in $O(u)$ and $f^n u \to p$.

LEMMA. *Suppose the condition* (Cδ) *holds on $O(x)$ for some regular point $x \in X$. Let $d_k = d_k(x)$ denote* diam $O(f^k x)$. *Then*

(i) $\lim_n d_n = 0$.

(ii) *If $f^k x = \lim_n f^n x$ for some $k \in \omega$, then $d_k = d_{k+1}$.*

Proof. (i) Since $d_{n+1} \leq d_n$ for all $n \in \omega$, $\{d_n\}$ converges to some $\varepsilon \geq 0$. Suppose $\varepsilon > 0$. Then there exists an $\varepsilon_0 < \varepsilon$ and a $\delta_0 > 0$ satisfying (Cδ). Choose $k \in \omega$ such that $\varepsilon \leq d_k < \varepsilon + \delta_0$. Then for $m \geq n \geq k$, we have

$$\varepsilon \leq d_n = \delta(f^m x,\ f^n x) \leq d_k < \varepsilon + \delta_0,$$

which implies $d(f^{m+1}x,\ f^{n+1}x) \leq \varepsilon_0$. This leads to $d_{k+1} \leq \varepsilon_0 < \varepsilon$, a contradiction. Therefore, $\varepsilon = 0$ and $\{f^n x\}$ is Cauchy.

(ii) Suppose $d_k \neq d_{k+1}$. Then we have $d_{k+1} < d_k$. Since

$$d_k = \max\{\sup\{d(f^m x,\ f^k x) \mid m > k\},\ d_{k+1}\},$$

we have

$$d_k = \sup\{d(f^m x,\ f^k x) \mid m > k\}.$$

For any $n > m > k$, we have

$$d(f^k x,\ f^m x) \leq d(f^k x,\ f^n)x + d(f^n x,\ f^m x)$$
$$\leq d(f^k x,\ f^n x) + d_{k+1}.$$

By letting $n \to \infty$, we have $d(f^k x,\ f^m x) \leq d_{k+1}$. This shows that $d_k \leq d_{k+1}$, a contradiction.

Proof of Theorem 2(Cδ). By (1) and the Lemma (i), $f^n u \to p$. Let $c_n = \sup\{d(f^m u,\ f^m p) \mid m \geq n\}$. Then $\{c_n\}$ is nonincreasing and $c_n \to \varepsilon$ for some $\varepsilon \geq 0$. Suppose $\varepsilon > 0$. Then there exists an $\varepsilon_0 < \varepsilon$ and a $\delta_0 > 0$ satisfying (Cδ) on $O(u) \cup O(p)$. Since $c_n \downarrow \varepsilon$, $d_n(u) \downarrow 0$, and $d_n(p) \downarrow 0$, there exists a $k \in \omega$ such that $n \geq k$ implies

$$c_n < \varepsilon + \delta_0/3,\quad d_n(u) < \delta_0/3,\quad d_n(p) < \delta_0/3.$$

Since

$$\varepsilon \leq c_n \leq \delta(f^n u,\ f^n p) \leq d_n(u) + c_n + d_n(p) < \varepsilon + \delta_0,$$

it follows that

$$d(f^{n+1} u,\ f^{n+1} p) \leq \varepsilon_0 < \varepsilon\ ;$$

that is, $c_{k+1} \leq \varepsilon_0 < \varepsilon$, a contradiction. Therefore, $\varepsilon = 0$. Hence, we have $d(f^n u,\ f^n p) \to 0$ as $n \to \infty$, and $f^n p \to p$. By the Lemma (ii) with $k=0$, we have $d_0(p) = d_1(p)$. Suppose $d_0(p) = d_1(p) = \ldots = d_k(p) > d_{k+1}(p)$ for some $k \geq 1$. Then, we have $d_k(p) = \sup\{d(f^m p,\ f^k p) \mid m > k\}$ as in the proof of Lemma (ii). For $\varepsilon = d_k(p)$ there exist $\varepsilon_0 < \varepsilon$ and $\delta_0 > 0$ satisfying (Cδ) on $O(p)$. Hence, for any $m > k$,

$$\varepsilon = \delta(f^{k-1} p,\ f^{m-1} p) = d_{k-1}(p) < \varepsilon + \delta_0$$

implies

$$d(f^k p,\ f^m p) \leq \varepsilon_0,$$

that is, $d_k(p) \leq \varepsilon_0 < \varepsilon$, a contradiction. Therefore, $d_0(p) = d_k(p)$ for all $k \in \omega$, and we have $d_0(p) = 0$ by Lemma (i). This shows that $fp = p$. The uniqueness is clear.

From Theorem 2(Cδ), we obtain

THEOREM 3(Cδ). *Let f be a selfmap of a complete metric space X. If (Cδ) holds for all regular points $x, y \in X$, then f has a unique fixed point $p \in X$, and $f^n x \to p$ for any regular point $x \in X$.*

Proof. Let x_0 be a regular point of X. Then, from the Lemma (ii), $\{f^n x_0\}_{n \in \omega}$ is a Cauchy sequence, and hence converges to some $p \in X$. The conclusion follows from Theorem 2(Cδ).

Theorem 3(Cδ) is given by Hegedüs–Szilágyi [12]. An easy example showing that Theorem 2(Cδ) properly extends Theorem 3(Cδ) is given in [25], [19].

Replacing the condition (Cδ) by other conditions we can deduce many consequences. Actually, Theorems 3(Cδ) and 3(Eδ) are given by Hegedüs–

Szilágyi [13] and Hegedüs [11] respectively, Theorems 2(Dδ) and 3(Dδ) by Kasahara [19], Theorem 3(Cδ) by Boyd-Wong [4]. Theorem 2(Dm) by Daneš [7], Husain-Sehgal [14], Theorem 3(Em) by Ćirić [5] and Massa [20], and Theorem 3(Dd) by Browder [4]. Note that Theorem 3(Bd) is given by Meir-Keeler [21] and Theorem 2(Bd) by Park [26].

Theorem 3(Eδ) of Hegedüs [11] is the following.

THEOREM 3(Eδ). *If f is a selfmap of a complete metric space X satisfying the condition* (Eδ), *and if every point is regular for f, then*

(1) *f has a unique fixed point $x^* \in X$,*

(2) *the Picard iteration of any $x \in X$ converges to x^*, and*

(3) *for any $x \in X$, we have*

$$d(x^*, f^n x) \leq \alpha^n d(x, fx)/(1-\alpha) \quad (n=0, 1, 2, \ldots),$$

$$d(x^*, f^n x) \leq \alpha\, d(f^{n-1}x, f^n x)/(1-\alpha) \quad (n=1, 2, 3, \ldots).$$

Some variants of Theorem 2(Cδ) are possible. One of the most popular type is the following.

COROLLARY 2(Cδ). *Let f be a selfmap of a metric space X. Suppose there exists a positive integer k such that f^k satisfies the hypothesis of Theorem 2* (Cδ). *Then f has a unique fixed point in $O(u)$.*

Similarly, we can state a Corollary 3(Eδ) such that the conclusion includes approximation formulas as in Ćirić [5].

4. For commuting maps

Jungck [17] first gave a fixed point theorem for commuting selfmaps f and g of a complete metric space X satisfying the conditions $gX \subset fX$, f is continuous, and

(Ed)′ $$d(gx, gy) \leq \alpha\, d(fx, fy), \qquad \alpha \in [0, 1).$$

Similarly, we can consider other conditions ( )′ just imitating (Ed)′. There have appeared a number of extensions of Jungck's theorem for other contractive type conditions; for example, Jeoung [16] for (Cd)′, Ranganathan [29] for (Em)′, Kasahara [18] and Park [25] for (Dm)′, Park-Rhoades [28] for (Dδ)′, and Bae-Park [1] for (Bd)′. A unified approach to these extensions is given in [27].

Now we show that Theorem 2(Cδ) can be extended to a theorem with respect to commuting maps.

Let f and g be selfmasp a metric space X. A point $x_0 \in X$ is said to be regular if there exists a sequence $\{x_n\}_{n\in\omega}$ in X such that $fx_{n+1} = gx_n$ for each $n \in \omega$ and diam $\{fx_n \mid n \in \omega\} < \infty$. The set $\{fx_n \mid n \in \omega\}$ will be denoted by $O(x_0)$.

THEOREM $2(\mathrm{C}\delta)'$. *Let f and g be commuting selfmaps of a metric space X. Suppose there exists a regular point $u_0 \in X$ such that*

(1) *$O(u_0)$ has a regular cluster point $p_0 \in X$, at which f is continuous, and*

(2) *the following condition holds:*

$(\mathrm{C}\delta)'$ *for any $\varepsilon > 0$, there exist $\varepsilon_0 < \varepsilon$ and $\delta_0 > 0$ such that for any $x, y \in \{x_n\} \cup \{p_n\} \cup \{fp_0\}$,*

$\varepsilon \leq \mathrm{diam}\,(O(x) \cup O(y)) < \varepsilon + \delta_0$ *implies $d(gx, gy) \leq \varepsilon_0$.*

Then $fp_0 = gp_0$ and $fu_n \to p_0$. If $(\mathrm{C}\delta)'$ holds for all regular points $x, y \in X$, then fp_0 is the unique common fixed point of f and g.

The proofs of Theorem $2(\mathrm{C}\delta)$ and the main result of Park-Rhoades [28] can be easily combined and modified to prove Theorem $2(\mathrm{C}\delta)'$.

For $f = 1_X$, the identity map of X, Theorem $2(\mathrm{C}\delta)'$ reduces to Theorem $2(\mathrm{C}\delta)$.

5. Problems

We conclude this paper by raising some open questions.

(1) Are there other counterexamples of the implications between various conditions in Section 2?

(2) Are there any extensions of Theorem $2(\mathrm{C}\delta)$ to the conditions (Bm), and (Bδ), or are they independent?

(3) C. S. Wong [33], [34] gave characterizations of the contractive type condition (Bd) of Meir-Keeler [21] and some others. Are there any similar characterizations of (Bm) and (Bδ), or of others?

(4) The conditions (Ad), (Am), and (Aδ) imply $d(fx, fy) \leq d(x, y)$ (nonexpansive) [10], $d(fx, fy) \leq m(x, y)$, and $d(fx, fy) \leq \delta(x, y)$, respectively. There are a number of theorems on nonexpansive maps of certain spaces. Can those be extended to more general maps satisfying $d(fx, fy) \leq \delta(x, y)$?

(5) Most of the results of this paper are also true in generalized metric spaces, L-spaces, Hausdorff uniform spaces, and 2-metric spaces, and are probably extendable to multi-valued functions.

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Seoul National University