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INFINITESIMAL VARIATIONS OF THE INVARIANT SUBMANIFOLD OF A P-SASAKIAN MANIFOLD

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§ 0. Introduction

In recent year, Sato ([1]) has introduced P-Sasakian structure (or normal paracontact Riemannian structure) and a number of authors has studied some characteristic properties of a P-Sasakian manifold ([1], [2], [3]).

On the other hand, many authors have studied infinitesimal variations of submanifold of Riemannian and Kaehlerian manifold. Moreover K. Yano, U-Hang Ki and J. S. Pak ([6]) proved that an infinitesimal fibre-preserving invariant conformal variation of a compact orientable invariant submanifold of a Sasakian manifold is necessarily f-preserving, where f-preserving means that it is invariant and it preserves the induced tensor field $f_a{}^b$ of type (1, 1) on the invariant submanifold of a Sasakian manifold. And K. Matsumoto has proved theorems analogous to those proved in ([6]) in the invariant hypersurfaces of a P-Sasakian manifold.

The purpose of the present paper is to study infinitesimal variations of a compact orientable submanifold of a P-Sasakian manifold and to prove theorems analogous to those proved in ([4], [6]). Thanks are due to Professor U-Hang Ki for his invaluable advice.

§ 1. Prelimsinaries

Let M^n be a n-dimensional P-Sasakian manifold covered by a system of coordinate neighberhoods $\{U; x^h\}$ and g_{ji} be the Riemannian metric where and in the sequel the indices $h, i, j, \dots$ run over the range $\{1, 2, \dots, n\}$.

Then we have

(1.1) $\nabla_j f_i - \nabla_i f_j = 0,$

(1.2) $\nabla_k \nabla_j f_i = (-g_{ki} + f_k f_i) f_j + (-g_{kj} + f_k f_j) f_i,$

f^i being a unit vector field of M^n and $f_i = f^j g_{ji}$ ([1], [2], [3]), where ∇_j denotes the Levi-Civita covariant differentiation. Now if we put

(1.3) $f_j{}^i = \nabla_j f^i$

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then we have

(1.4) $f_j f^j = 1$, $f_j^{\ i} f^j = 0$, $f_i f_j^{\ i} = 0$,

(1.5) $f_j^{\ i} f_i^{\ h} = \delta_j^h - f_j f^h$,

(1.6) $f_{ji} = f_{ij}$ $(f_{ji} = g_{it} f_j^{\ t})$,

and

(1.7) $g_{st} f_j^{\ s} f_i^{\ t} = g_{ji} - f_j f_i$.

Then we can easily obtain

(1.8) $K_{kji}{}^h f_h = g_{ki} f_j - g_{ji} f_k$,

(1.9) $K_{ji} f^i = -(n-1) f_j$,

(1.10) $K_{jt} f_i^{\ t} - K_{jtsi} f^{ts} = (n-2) f_{ji} - \bar{\phi} g_{ji} + 2\bar{\phi} f_j f_i$,

(1.11) $K_{jt} f_i^{\ t} = K_{it} f_j^{\ t}$,

where $K_{kji}{}^h$ and K_{ji} are respectively the curvature tensor and the Ricci tensor with respect to g_{ji}, $\bar{\phi}$ is defined as $\bar{\phi} = f_{ji} g^{ji}$ ([3]).

Let M^m be a m-dimensional Riemannian manifold isometrically immersed in M^n by the isometric immersion $i: M^m \to M^n$ and covered by the local coordinate system $\{V; y^a\}$. We identify $p \in M^m$ with $i(p) \in M^n$ and the tangent space $T_p M^m$ with a subspace of $T_p M^n$. In terms of local coordinates (y^a) of M^m and (x^h) of M^n the immersion i is locally expressed by $x^h = x^h(y^a)$.

If we put $B_a^{\ i} = \partial_a x^i$, $\partial_a = \partial/\partial y^a$, then $B_a^{\ i}$ are m-linearly independent vectors of M^n tangent to M^m. Denote by g_{ba} the Riemannian metric tensor of M^m, we have

$$g_{cb} = B_c^{\ j} B_b^{\ i} g_{ji}$$

because the immersion is isometric.

We denote by $C_y^{\ h}$ $(n-m)$ mutually unit normals to M^m. Then the metric tensor of the normal bundle of M^m is given by $g_{yx} = C_y^{\ j} C_x^{\ i} = \delta_{yx}$, δ_{yx} denoting the Kronecker delta. The systems of indices $a, b, c, \ldots$ and $x, y, z, \ldots$ run over the ranges $\{1, 2, \ldots, m\}$ and $\{m+1, \ldots, n\}$ respectively and the summation convension will be used with respect to these indices.

Let $h_{ba}{}^x$ be second fundamental tensors of M^m. Then we have the following Gauss and Weingarten equations

(1.12) $\nabla_b B_a^{\ i} = h_{ba}{}^x C_x^{\ i}$, $\nabla_b C_x^{\ i} = -h_b{}^a{}_x B_a^{\ i}$

∇_b being the so-called van der Waerden-Bortolotti covariant differentation and $h_b{}^a{}_x = h_{be}{}^y g^{ea} g_{yx}$, where $\nabla_b B_a^{\ i}$, $\nabla_b C_x^{\ i}$ are

$$\nabla_b B_a^{\ i} = \partial_b B_a^{\ i} - \left\{ {c \atop b\ a} \right\} B_c^{\ i} + \left\{ {i \atop j\ k} \right\} B_b^{\ j} B_a^{\ k},$$

$$\nabla_b C_x^{\ i} = \partial_b C_x^{\ i} + \left\{ {i \atop j\ k} \right\} B_b^{\ j} C_x^{\ k} - \Gamma_b{}^y{}_x C_y^{\ i}$$

and $\left\{ {i \atop j\ k} \right\}$ and $\left\{ {a \atop b\ c} \right\}$ are the Christoffel's symbols formed with g_{ji} and g_{cb} respectively and $\Gamma_b{}^y{}_x$ are the components of the connection induced on the normal bundle of M^m from the Riemannian connection ∇ of M^n, that is,

$$\Gamma_b{}^y{}_x = \left(\partial_b C_x{}^i + \left\{ {i \atop j\ k} \right\} B_b{}^j C_x{}^k\right) C^y{}_i, \quad C^y{}_i = C_x{}^h g^{xy} g_{hi}.$$

Denoting $K_{dcb}{}^a$ and $K_{dcy}{}^x$ the curvature tensors of M^m and of the normal bundle of M^m, we have the following structure equations of Gauss, Codazzi and Ricci respectively:

(1.13) $K_{dcb}{}^a = K_{kji}{}^h B_d{}^k B_c{}^j B_b{}^i B_h{}^a + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x,$

(1.14) $0 = K_{kji}{}^h B_c{}^k B_b{}^j B_a{}^i C_h{}^x - (\nabla_c h_{ba}{}^x - \nabla_b h_{ca}{}^x),$

and

(1.15) $K_{dcy}{}^x = K_{kji}{}^h B_d{}^k B_c{}^j C_y{}^i C_h{}^x + h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y.$

A m-dimensional submanifold M^m of M^n is called invariant (or an invariant submanifold) when each tengent space of M^m is invariant under the action of $f_j{}^i$. Hence in this case, we can put

(1.16) $f_j{}^i B_b{}^j = f_b{}^a B_a{}^i, \quad f_j{}^i C_x{}^j = f_x{}^y C_y{}^i,$

$f_b{}^a$ and $f_y{}^x$ being tensor fields of type (1, 1) of M^m and the normal bundle of M^m respectively. Putting $f_{ba} = f_b{}^e g_{ea}$, $f_{yx} = f_y{}^z g_{zx}$, we have

$$f_{ba} = f_{ab}, \quad f_{yx} = f_{xy}.$$

On the other hand, we put

(1.17) $f^i = f^a B_a{}^i + f^x C_x{}^i.$

Transvecting the equations of (1.16) with $f_i{}^j$ and making use of (1.5), (1.16) and (1.17), we have

(1.18) $f_b{}^a f_a{}^e = \delta_b^e - f_b f^e, \quad f_b f^x = 0,$

(1.19) $f_x{}^z f_z{}^y = \delta_x^y - f_x f^y.$

And now from (1.4) we have

(1.20) $f_a f^a + f_x f^x = 1,$

(1.21) $f^a f_a{}^b = 0, \quad f^x f_x{}^y = 0.$

Differentiating covariantly the equations of (1.16) and making use of (1.2), (1.12), (1.16), (1.17) and (1.18), we have

(1.22) $\Delta_c f_b{}^a = (-\delta_c^a + f_c f^a) f_b + (-g_{cb} + f_c f_b) f^a,$

(1.23) $h_{ca}{}^x f_b{}^a = h_{cb}{}^y f_y{}^x - g_{cb} f^x,$

(1.24) $\nabla_c f_x{}^y = 0.$

Finally differentiating covariantly the equation (1.17) and making use of (1.3), (1.12) and (1.16), we can obtain

(1.25) $\nabla_b f^a = f_b{}^a + f^x h_b{}^a{}_x,$

(1.26) $\nabla_b f^x = -f^a h_{ba}{}^x.$

From the last equation of (1. 18) and (1. 20), we can see that there exist only two cases: (1) $f^x=0$ or (2) $f_b=0$.

(1) In case $f^x=0$, that is,	(2) In case $f_b=0$, that is,
the P-Sasakian structure vector f^i of the ambient manifold M^n is tangent	the P-Sasakian structure vector f^i of the ambient manifold M^n is normal to the submanifold M^m,
(1. 18)$\sim$(1. 26) reduce	(1. 18)$\sim$(1. 26) reduce
(1. 27) $f_b{}^a f_a{}^e=\delta_b^e-f_b f^e$	(1. 27)′ $f_b{}^a f_a{}^e=\delta_b^e$
(1. 28) $f_x{}^z f_z{}^y=\delta_x^y$	(1. 28)′ $f_x{}^z f_z{}^y=\delta_x^y-f_x f^y$
(1. 29) $f_a f^a=1$	(1. 29)′ $f_x f^x=1$
(1. 30) $f^a f_a{}^b=0$	(1. 30)′ $f_x f_y{}^x=0$
(1. 31) $\nabla_c f_b{}^a=-\delta_c^a f_b+2f_c f_b f^a - g_{cb} f^a$	(1. 31)′ $\nabla_c f_b{}^a=0$
(1. 32) $h_{ca}{}^x f_b{}^a=h_{cb}{}^y f_y{}^x$	(1. 32)′ $h_{ca}{}^x f_b{}^a=h_{cb}{}^y f_y{}^x-g_{cb} f^x$
(1. 33) $\nabla_c f_x{}^y=0$	(1. 33)′ $\nabla_c f_x{}^y=0$
(1. 34) $\nabla_b f^a=f_b{}^a$	(1. 34)′ $f_b{}^a=-f^x h_b{}^a{}_x$
(1. 35) $f^a h_{ba}{}^x=0.$	(1. 35)′ $\nabla_b f^x=0.$
(1. 27), (1. 29), (1. 30), (1. 31) and (1. 34) show that $(f_b{}^a,\ g_{cb},\ f_b)$ admits a P-Sasakian structure in M^m.	(1. 27)′ and (1. 31)′ show that $(f_b{}^a, g_{cb})$ admits an almost product structure in M^m,

Thus we have

(1. 36) $K_{dcb}{}^a f_a=-f_d g_{cb}+f_c g_{db}$

(1. 37) $K_{cb} f^b=-(m-1) f_c$

(1. 38) $K_{ce} f_b{}^e-K_{ceab} f^{ea} =(m-2)f_{cb}-\phi g_{cb}+2\phi f_c f_b$

(1. 39) $K_{ce} f_b{}^e=K_{be} f_c{}^e$, where $\phi=f_{cb}g^{cb}$.

§ 2. Infinitesimal variations of invariant submanifolds

We consider an infinitesimal variation of invariant submanifold M^m of M^n given by

(2. 1) $\bar{x}^h=x^h(y)+\xi^h(y)\varepsilon,$

where $\xi^h(y)$ is a vector field of M^n defined along M^m and ε is an infinitesimal. We then have

(2. 2) $\bar{B}_b{}^h=B_b{}^h+(\partial_b\xi^h)\varepsilon,$

where $\bar{B}_b{}^h=\partial_b\bar{x}^h$ are m-linearly independent vectors tangent to the varied submanifold. We displace $\bar{B}_b{}^h$ parallelly from the varied point $(\bar{x}^h)$ to the original point (x^h). Then we obtain the vectors

$$\tilde{B}_b{}^h=\bar{B}_b{}^h+\Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j\bar{B}_b{}^i\varepsilon$$

at the point (x^h), or $\tilde{B}_b{}^h=B_b{}^h+(\nabla_b\xi^h)\varepsilon$, neglecting the terms of order higher than one with respect to ε, where

(2.3) $\nabla_b\xi^h=\partial_b\xi^h+\Gamma_{ji}{}^hB_b{}^i\xi^j.$

In the equel we always neglect terms of order higher than one with respect to ε. Thus putting $\delta B_b{}^h=\tilde{B}_b{}^h-B_b{}^h$, we have $\delta B_b{}^h=(\nabla_b\xi^h)\varepsilon$.

If we put

(2.4) $\xi^h=\xi^aB_a{}^h+\xi^xC_x{}^h,$

then we obtain

(2.5) $\nabla_b\xi^h=(\nabla_b\xi^a-h_{bx}{}^a\xi^x)B_a{}^h+(\nabla_b\xi^x+h_{ba}{}^x\xi^a)C_x{}^h$

because of (1.12).

Now we denote by $\bar{C}_y{}^h$ (n–m) mutually orthogonal unit normals to the varied submanifold and by $\tilde{C}_y{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $\bar{C}_y{}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

(2.6) $\tilde{C}_y{}^h=\bar{C}_y{}^h+\Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j\bar{C}_y{}^i\varepsilon.$

We put

(2.7) $\delta C_y{}^h=\tilde{C}_y{}^h-C_y{}^h$

and assume that $\delta C_y{}^h$ is of the form

(2.8) $\delta C_y{}^h=\eta_y{}^h\varepsilon=(\eta_y{}^aB_a{}^h+\eta_y{}^xC_x{}^h)\varepsilon.$

Then, from (2.6), (2.7) and (2.8), we have

(2.9) $\bar{C}_y{}^h=C_y{}^h-\Gamma_{ij}{}^h\xi^jC_y{}^i\varepsilon+(\eta_y{}^aB_a{}^h+\eta_y{}^xC_x{}^h)\varepsilon.$

Applying the operator δ to $B_b{}^jC_y{}^ig_{ji}=0$ and using (2.5), (2.8) and $\delta g_{ji}=0$, we find

$$(\nabla_b\xi_y+h_{bay}\xi^a)+\eta_{yb}=0,$$

where $\xi_y=\xi^zg_{zy}$ and $\eta_{yb}=\eta_y{}^cg_{cb}$, or

(2.10) $\eta_y{}^a=-(\nabla^a\xi_y+h_b{}^a{}_y\xi^b),$

∇^a being defined to be $\nabla^a=g^{ac}\nabla_c$. Applying also the operator δ to $C_y{}^jC_z{}^ig_{ji}=g_{yz}$ and using (2.8) and $\delta g_{ji}=0$, we find

(2.11) $\eta_{yx}+\eta_{xy}=0,$

where $\eta_{yx}=\eta_y{}^zg_{zx}$.

We assume that the infinitesimal variation (2.1) carries an invariant submanifold into an invariant submanifold, that is,

(2.12) $f_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i$ are linear combination of $\bar{B}_b{}^h$.

Now using the equations of (1.2), (1.16), (1.17), (1.18), (2.2), (2.3), (2.4) and (2.5), we have

$$(2.13)\quad f_i^h(x+\xi\varepsilon)\bar{B}_b^i=[f_b^a-f_b^e(\nabla_e\xi^a-h_e{}^a{}_x\xi^x)\varepsilon+f_e^a(\nabla_b\xi^e-h_b{}^e{}_x\xi^x)\varepsilon+2(\xi^e f_e)f_b f^a\varepsilon-f_b\xi^a\varepsilon-f^a\xi_b\varepsilon]\bar{B}_b^h+[f_y{}^x(\nabla_b\xi^y+h_{ba}{}^y\xi^a)-f_b^a(\nabla_a\xi^x+h_{ae}{}^x\xi^e)-f_b\xi^x-\xi_b f^x]\bar{C}_x^h\varepsilon.$$

Thus (2.12) is equivalent to

$$(2.14)\quad f_y{}^x(\nabla_b\xi^y)-f_b^a(\nabla_a\xi^x)-f_b\xi^x=0$$

by the virtue of (1.25).

An infinitesimal variation given by (2.1) is called an *invariant* (or *invariance-preserving*) *variation* if it carries an invariant submanifold into an invariant submanifold. When $\xi^x=0$, that is, when the variation vector ξ^h is tangent to the submanifold, the variation is said to be *tangential* and when $\xi^a=0$, that is, when the variation vector ξ^h is normal to the submanifold, the variation is said to be *normal*. When the tangent space at a point (x^h) of the submanifold and that at the corresponding point $(\bar{x}^h)$ of the varied submanifold are always parallel, the variation is said to be *parallel*. Then we have the following assertions:

LEMMA 2.1. *In order for an infinitesimal variation to be invariant, it is necessary and sufficient that the variation vector satisfies* (2.14).

LEMMA 2.2. *In order for an infinitesimal variation to be parallel, it is necessary and sufficient that*

$$\nabla_b\xi^x+h_{ba}{}^x\xi^a=0.$$

LEMMA 2.3. *If an infinitesimal invariant variation of M^m is parallel, then it is tangential in case* (1), *that is, it is tangential in the case that the vector fields f^i are always tangent to the submanifold, and it is normal in case* (2), *that is, it is normal in the case that the vector fields f^i are always normal to the submanifold.*

§3. The variations of f_b^a

Suppose that an infinitesimal variation (2.1) is invariant. Then putting

$$(3.1)\quad f_i^h(x+\xi\varepsilon)\bar{B}_b^i=(f_b^a+\delta f_b^a)\bar{B}_a^h,$$

we have from (1.15) and (2.15)

$$(3.2)\quad \delta f_b^a=[f_e^a\nabla_b\xi^e-f_b^e\nabla_e\xi^a+2(\xi^e f_e)f_b f^a-f_b\xi^a-f^a\xi_b]\varepsilon.$$

If an invariant variation preserves f_b^a, then we say that it is *f-preserving*.

LEMMA 3.1. *An invariant variation is f-preserving if and only if*

$$(3.3)\quad (\nabla_b\xi^e)f_e^a-f_b^e(\nabla_e\xi^a)+2(\xi^e f_e)f_b f^a-f_b\xi^a-\xi_b f^a=0.$$

Now applying the operator δ to $g_{cb}=g_{ji}B_c^jB_b^i$, and using $\delta g_{ji}=0$, we find

(3.4) $\delta g_{cb}=(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cb}{}^x\xi_x)\varepsilon,$

from which we have

$$\delta g^{cb}=-(\nabla^b\xi^c+\nabla^c\xi^b-2h^{cb}{}_x\xi^x)\varepsilon.$$

A variation of a submanifold for which $\delta g_{cb}=0$ is said to be *isometric* and for which δg_{cb} is proportional to g_{cb} is said to be *conformal*. A necessary and sufficient condition for an infinitesimal variation (2.1) of a submanifold to be conformal is

(3.5) $\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x=2Ag_{cb},$

where $A=(1/m)(\nabla_e\xi^e-h_e{}^e{}_x\xi^x)$.

Since the infinitesimal variation (2.1) is invariant, we have

(3.6) $\bar{f}_i{}^h\bar{C}_y{}^i=\bar{f}_y{}^x\bar{C}_x{}^h.$

Then using (2.9), we find

(3.7) $f_i{}^h(x+\xi\varepsilon)[C_y{}^i-\Gamma_{jt}{}^i\xi^jC_y{}^t\varepsilon+(\eta_y{}^aB_a{}^i+\eta_y{}^xC_x{}^i)\varepsilon]=(f_y{}^x+\delta f_y{}^x)\bar{C}_x{}^h$

from which we can get

(3.8) $\eta_y{}^ef_e{}^a-\xi^af_y-\xi_yf^a=f_y{}^x\eta_x{}^a,$

(3.9) $\delta f_y{}^z=[-f_y{}^x\eta_x{}^z+\eta_y{}^xf_x{}^z-f^z\xi_y-f_y\xi^z+2(\xi^xf_x)f_yf^z]\varepsilon.$

On the other hand, applying the operator δ to (1.29) and (1.30), we have the variations of f^c in case (1) by the help of (3.2) and (3.4)

(3.10) $\delta f^c=\mathfrak{L}f^c\varepsilon,$

$\mathfrak{L}$ being the operator of the Lie derivation.

We now define a tensor field T_{cb} by

(3.11) $T_{cb}=\nabla_c\xi_b-(\nabla_e\xi_d)f_c{}^ef_b{}^d-(f_b{}^e\xi_e)f_c{}'+(f_c{}^e\xi_e)f_b-f^ef^d(\nabla_e\xi_d)f_cf_b,$

(3.11)′ $T_{cb}=\nabla_c\xi_b-(\nabla_e\xi_a)f_c{}^ef_b{}^a$

for the case (1) and (2) respectively, and prove

LEMMA 3.2. *In order for an infinitesimal invariant variation of an invariant submanifold to be f-preserving, it is necessary and sufficient that $T_{cb}=0$.*

Proof. Suppose that an infinitesimal invariant variation of an invariant submanifold is f-preserving. Then in case (1) by Lemma 3.1, we have

$$V_b\xi^c-f_e(V_b\xi^e)f^c-(V_e\xi^a)f_a{}^cf_b{}^e-f_b\xi^ef_e{}^c=0,$$

by transvecting (3.3) with $f_a{}^c$ and using (1.27) and (1.30),

$$f_aV_d\xi^a=f^ef_a(V_e\xi^a)f_d-f_d{}^e\xi_e$$

by transvecting (3.3) with $f_af_d{}^b$ and using (1.29) and (1.30) respectively. These two equations imply that $T_{cb}=0$.

Conversely we suppose that $T_{cb}=0$. Then we have by transvecting (3.11) with f^c

$$f^eV_b\xi_e=f^ef^a(\nabla_e\xi_a)f_b-f_b{}^e\xi_e,\quad f^e\nabla_e\xi_b=f^ef^a(\nabla_e\xi_a)f_b+f_b{}^e\xi_e.$$

Transvecting (3. 11) with $f_a{}^c$ and taking account of (1. 27), (1. 30) and the last equations, we have our lemma. In case (2), from Lemma 3. 1 and (1. 27)′ we can easily verify our lemma.

And consequently we shall prove

LEMMA 3. 3 *For an infinitesimal conformal invariant variation of an invariant minimal submanifold, we have;*

In case (1)	*In case* (2)
(3. 12) $T_{cb}+T_{bc}=0$	(3. 12)′ $T_{cb}+T_{bc}=0$
(3. 13) $T_{cb}+f_c{}^e f_b{}^a T_{ea}=(\mathfrak{L}f_c)f_b-(\mathfrak{L}f_b)f_c$	(3. 13)′ $T_{cb}+f_c{}^e f_b{}^a T_{ea}=0$
(3. 14) $T_{cb}T^{cb}=2T^{cb}\nabla_c\xi_b+4(\mathfrak{L}f_c)f_e{}^c\xi^e-2(\mathfrak{L}f_c)(\mathfrak{L}f^c)-2A(\mathfrak{L}f_c)f^c.$	(3. 14)′ $T_{cb}T^{cb}=2T^{cb}\nabla_c\xi_b.$

Proof. Differentiating (3. 5) covariantly and using Ricci identity and the first Biahchi identity, we find

$$(3.15)\quad \nabla_c\nabla_b\xi_a-K_{bacd}\xi^d=\nabla_c(h_{ba}{}^x\xi_x+Ag_{ba})+\nabla_b(h_{ca}{}^x\xi_x+Ag_{ca})-\nabla_a(h_{cb}{}^x\xi_x+Ag_{cb}).$$

Transvecting this with g^{cb} and using the condition $h_e{}^e{}_x=0$, that is, the submanifold is *minimal*, we find

$$(3.16)\quad \nabla^c\nabla_c\xi_a+K_{ac}\xi^c-2\nabla^c(h_{ca}{}^x\xi_x)+(m-2)\nabla_aA=0.$$

On the other hand, in case (1) we can easily verify that

$$(3.17)\quad f^cf^b(\nabla_c\xi_b)=A,$$

by the virtue of (3. 5).

Thus from the definition (3. 13), (1. 28) and (1. 32), we see that (3. 12).

Now using (1. 27) and (3. 17), we have

$$f_d{}^c f_b{}^e T_{ec}=f_d{}^c f_b{}^e(\nabla_e\xi)-\nabla_b\xi_d-\xi^e f_{eb}f_d+\xi^e f_{ed}f_b+f_d\mathfrak{L}f_b-f_b\mathfrak{L}f_d+Af_bf_d$$

where we used the identity $\mathfrak{L}f_b=\xi^e f_{eb}+f_e\nabla_b\xi^e$.

Consequently from (3. 11) and the last equations, we have (3. 13).

Next, from (3. 11) we can get

$$T_{cb}T^{cb}=T^{cb}\nabla_c\xi_b-AT^{cb}f_cf_b+T^{cb}f_c{}^e\xi_ef_b-T^{cb}f_b{}^e\xi_ef_c-T^{cb}f_c{}^ef_b{}^a(\nabla_e\xi_a).$$

Therefore using (3. 13) and $f^eT_{ec}=-\mathfrak{L}f_c+Af_c$, we have (3. 14)

Finall, in case (2), we can obtain our lemma from (3. 11)′ by the virtues of (3. 5), (1. 27)′, (1. 28)′ and (1. 32)′.

Applying the operator ∇^c to (3. 11), using (1. 38), (3. 16) and Ricci identity, and assumming that the submanifold is minimal, we then have

$$(3.18)\quad \nabla^eT_{eb}=-m\nabla_bA+A\phi f_b+\phi f_b{}^d\nabla_dA+f^e(\nabla_eA)f_b-(m-3)f_b{}^d\mathfrak{L}f_d.$$

Now an infinitesimal variation which satisfies $\mathfrak{L}f^b=@f^b$, @ being a certain function, is to be said to be *fibre-preserving*. Thus for a fibre-preserving variation from $\delta g_{cb}=2Ag_{cb}$ and (1. 29) we see that $\mathfrak{L}f^a=-Af^a$ by (3. 12). Putting $\mu=\xi_e f^e$, differentiating this covariantly and using the identity $\mathfrak{L}f_b=\xi^e f_{eb}+f_e\nabla_b\xi^e$ we then have

$$\nabla_d\mu=\mathfrak{L}f_d=Af_d.$$

Thus we obtain $\nabla_c\nabla_d\mu=(\nabla_cA)f_d+Af_{dc}$, from which we have

$$f^e(\nabla_eA)f_d=\nabla_dA,\ f_b{}^d(\nabla_dA)=0,$$

where we used Ricci identity and $f_{ab}=f_{ba}$.

Substituting this result into (3. 18) we have

$$\nabla^eT_{eb}=-(m-1)\nabla_bA+A\phi f_b,$$

from which we have

$$\nabla^e([T_{eb}+mAg_{eb}-Af_ef_b]\xi^b)=\frac{1}{2}T^{cb}T_{cb}+(m^2-1)A^2.$$

Thus if the submanifold is compact orientable, we have

$$\int[T^{cb}T_{cb}+2(m^2-1)A^2]dV=0,$$

dV being the volume element of M^m. Hence we have ([4], [6])

THEOREM 3. 5. *If an infinitesimal conformal invariant variation of a compact orientable invariant minimal submanifold of a P-Sasakian manifold whose structure vector f^i is tangent to the submanifold is fibre-preserving, then it is isometric and f-preserving.*

Similarly we have

THEOREM 3. 6. *For an infinitesimal conformal invariant variation of a compact orientable invariant minimal submanifold of a P-Sasakian manifold whose structure vector f^i is normal to the submanifold, if the ambient manifold is space of a constant curvature, then the variation is isometric and f-preserving. Moreover it is normal.*

Proof. If the submanifold is minimal, from (1. 32)′ and (1. 34)′ we can easily find $h_{cb}{}^x f^{ca}=-mf^x$ and $\phi=0$.

On the other hand, we assume that the ambient manifold is space of constant curvature, then we see that from (1. 13) and (1. 14),

$$K_{dcba}\xi^a=-(\xi_dg_{cb}-\xi_cg_{db})+h_{da}{}^x\xi^ah_{cbx}-h_{cax}\xi^ah_{db}{}^x,$$
$$\nabla_dh_{cb}{}^x=\nabla_ch_{db}{}^x.$$

Now applying the operator ∇^c to (3. 13)′, using (3. 17) and the last equations, and computing by straighforward, we have

$$\nabla^eT_{ec}=\nabla^e\nabla_e\xi_c+(m+1)\xi_c-h_{ec}{}^xh_a{}^e{}_x\xi^a-2\nabla_cA-2\nabla_eh_c{}^{ex}\xi_x.$$

Consequently by substituting (3. 16) into this equation, we have

$$\nabla^e T_{ec}=2m\xi_c-m\nabla_c A.$$

Hence we can obtian

$$\nabla^e[(T_{ec}+mAg_{ec})\xi^c]=2m\xi_c\xi^c+\frac{1}{2}T^{cb}T_{cb}+(mA)^2,$$

from which, if the submanifold is compact orientable, we have

$$\int[2m\xi_c\xi^c+\frac{1}{2}T^{cb}T_{cb}+(mA)^2]dV=0.$$

This completes the proof of our theorem.

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