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ON FINITE GROUPS WITH QUASI-DIHEDRAL SYLOW 2-GROUPS

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1. Introduction

A finite group S_{m+2} of order 2^{m+2}, $m\geq 2$, defined by

$$S_{m+2}=\langle x, y \mid x^{2^{m+1}}=y^2=1,\ \ x^y=x^{-1+2^m}\rangle$$

is called a *quasi-dihedral* group. The only finite simple groups known with quasi-dihedral Sylow 2-groups are

$$L_3(q)=PSL(3, q),\ \ q\equiv -1 \pmod 4,$$
$$U_3(q)=PSU(3, q),\ \ q\equiv 1 \pmod 4,$$
$$M_{11},$$

where M_{11} denotes the Mathieu group on 11 letters. It is also well known that Sylow 2-groups of

$$GL(2, q),\ \ q\equiv -1 \pmod 4,$$
$$GU(2, q),\ \ q\equiv 1 \pmod 4)$$

are quasi-dihedral.

In this paper, we will analyze the fusion of 2-elements for arbitrary finite groups with quasi-dihedral Sylow 2-groups, and give a *detailed* proof of the following Theorem (*cf.* [1]).

THEOREM *Let G be a finite group with a quasi-dihedral Sylow 2-group S and let T, Q be representatives of the conjugacy classes of four subgoups and quaternion subgroups respectively of S. Then one of the following holds*:

(1) *G has no normal subgroups of index 2, G has one conjugacy class of involutions and one of elements of order 4, $|N_G(T) : G_G(T)|=6$ and $|N_G(Q) : QC_G(Q)|=6$.*

(2) *G has a normal subgroup K of index 2 with dihedral Sylow 2-groups, K has no normal subgroups of index 2, G has one conjugacy class of involutions and two of elements of order 4, $|N_G(T) : C_G(T)|=6$ and $|N_G(Q) : QC_G(Q)|=2$.*

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(3) *G has a normal subgroup K of index 2 with generalized quaternion Sylow 2-groups, K has no normal subgroups of index 2, $Z(S)$ is weakly closed in S with respect to G, G has two conjugacy classes of involutions and one of elements of order 4, $|N_G(T):C_G(T)|=2$ and $|N_G(Q):QC_G(Q)|=6$.*

(4) *G has a normal 2-complement, G has two conjugacy classes of involutions and two of elements of order 4, $|N_G(T):C_G(T)|=2$ and $|N_G(Q):QC_G(Q)|=2$.*

The focal subgroup theorem [4] and Grün's theorem [3] will be used in the proof of our Theorem. In section 2, we will prove a large number of basic properties of quasi-dihedral groups. Some parts of these are needed for the proof of our Theorem.

The terminology and the notation in this paper are standard, and they are taken from [2]. All groups in this paper are assumed to be finite.

2. The quasi-dihedral group

The quasi-dihedral group S_{m+2} has the following properties.

LEMMA *Let $S=S_{m+2}=<x, y \mid x^{2^{m+1}}=y^2=1,\ x^y=x^{-1+2^m}>$ be a quasi-dihedral group of order $2^{m+2}, m\geq 2$, Then the following hold:*

(i) *S has exactly three maximal subgroups. They are*

$$H=\langle x\rangle,\quad \langle x^2, y\rangle \text{ and } \langle x^2, xy\rangle$$

which are cyclic, dihedral and generalized quaternion, respectively.

(ii) *The involutions in S are $z=x^{2^m}$ and x^iy, i even. The elements of order 4 in S are $x^{2^{m-1}}$, $zx^{2^{m-1}}$ and x^iy, i odd.*

(iii) *$S'=\Phi(S)=\langle x^2\rangle$ and S/S' is elementary abelian.*

(iv) *$Z(S)=\langle z\rangle$ and $S/Z(S)$ is dihedral.*

(v) *$\Omega_1(S)=\langle x^2, y\rangle$, $\Omega_1(S')=Z(S)$ and $\Omega_2(S')=\langle x^{2^{m-1}}\rangle$.*

(vi) *S has class $m+1$, and is of maximal class.*

(vii) *S has two conjugacy classes of involutions, represented by z and y, respectively. S has two conjugacy classes of elements of order 4, represented by $x^{2^{m-1}}$ and xy, respectively.*

(viii) *S has one conjugacy class of four subgroups, represented by $T=\langle z, y\rangle$. S has one conjugacy class of quaternionsubgroups, represented by $Q=\langle x^{2^{m-1}}, xy\rangle$. Moreover, we have*

$$C_S(T)=T,\quad |N_S(T):C_S(T)|=2,\quad C_S(Q)=Z(Q),\quad |N_S(Q):Q|=2.$$

(ix) *S has two conjugacy classes of cyclic subgroups of order 4, represented by $T_1=\langle x^{2^{m-1}}\rangle$ and $T_2=\langle xy\rangle$. We have $Q=\langle T_1, T_2\rangle$,* and

$$C_S(T_1)=H,\quad N_S(T_1)=S,\quad N_S(T_2)=Q,\quad C_S(T_2)=T_2.$$

In particular, $|N_S(T_i) : C_S(T_i)|=2$, $1\leq i\leq 2$.

(x) *Any abelian subgroup of S of order at least 8 is contained in H and is cyclic.*

(xi) *Any proper normal subgroup of S is either maximal or is cyclic and contained in S'.*

(xii) *If D is a dihedral subgroup of S of order at least 8, then the maximal cyclic subgroup of D is contained in the maximal cyclic subgroup H.*

(xiii) $\mathrm{Aut}(S)$ *is a nonabelian 2-group of order* 2^{2m}.

Proof. Since S is generated by two elements, it follows from Burnside's basis theorem that $S/\Phi(S)$ is an elementary abelian group of order 4. Hence there are exactly three maximal subgroups, and they are $H=\langle x\rangle$, $\langle x^2, y\rangle$ and $\langle x^2, xy\rangle$. And it is easy to show that $\langle x\rangle$, $\langle x^2, y\rangle$ and $\langle x^2, xy\rangle$ are cyclic, dihedral and generalized quaternion of order 2^{m+1}, respectively. Thus (i) holds.

The only involution in H is $z=x^{2^m}$, and the only elements of order 4 in H are $x^{2^{m-1}}$ and $x^{-2^{m-1}}=zx^{2^{m-1}}$. On the other hand, each element in $S-H$ is of the form x^iy. Since we have $(x^iy)^2=z^i$ and $(x^iy)^4=1$, the order of x^iy is either 2 or 4, and x^iy is an involution if and only if i is even. Thus (ii) holds.

Next, (iii)$\sim$(vi) have been proved in Thorem 5.4.3 of [2].

The involution z is conjugate only to itself in S. Since $C_S(y)=\langle z, y\rangle=T$, the size of the conjugacy class containing y is $|S : T|=2^m$. Hence y is conjugate in S to x^iy for all even integer i. Thus S has two conjugacy classes of involutions. Similarly, the element $x^{2^{m-1}}$ is conjugate to itself and $x^{-2^{m-1}}$ in S. Since $C_S(xy)=\langle xy\rangle=T_2$, the size of the conjugacy class in S containing xy is $|S : T_2|=2^m$. Hence xy is conjugate in S to x^iy for all odd integer i. In particular, S has two conjugacy classes of elements of order 4. This proves (vii).

Let A be an arbitrary four subgroup of S, and let B be an arbitrary quaternion subgroup of S. Since $|S : H|=2$, we have $|H\cap A|=2$ and $|H\cap B|=4$, whence $H\cap A=\langle z\rangle=Z(S)$ and $H\cap B=\langle x^{2^{m-1}}\rangle=T_1$. Hence it follows from (ii) that $A=\langle z, x^iy\rangle$ for some even integer i and $B=\langle x^{2^{m-1}}, x^jy\rangle$ for some odd integer j. On the other hand, there exist elements u and v in S such that $y^u=x^iy$ and $(xy)^v=x^jy$ by (vii). Since u centralizes z and v normalizes T_1. it follows that $A=\langle z, y\rangle^u=T^u$ and $B=\langle x^{2^{m-1}}, xy\rangle^v=Q^v$. Thus every four subgroup of S is conjugate to T, and every quaternion subgroup of S is conjugate to Q.

There are exactly 2^m involutions which are not z, so the preceding result implies that there are exactly 2^{m-1} four subgroups of S. Hence it follows that $|S:N_S(T)|=2^{m-1}$. Moreover, $C_S(T)=C_S(y)=T$. Therefore, we have $|N_S(T):C_S(T)|=2$. On the other hand, there are exactly 2^{m-1} subgroups of S of order 4 which are not T_1. Hence the preceding result implies that there are exactly 2^{m-2} quaternion subgroups of S, so we have $|S:N_S(Q)|=2^{m-2}$. This yields that $|N_S(Q)|=2^4$ and $|N_S(Q):Q|=2$. It is clear that $C_S(Q)=C_H(xy)=\langle z\rangle=Z(Q)$. Thus (viii) holds.

The first part of (ix) follows from (vii). There are exactly 2^{m-1} subgroups which are conjugate to T_2 in S. Hence $|S:N_S(T_2)|=2^{m-1}$. Since $Q\subseteq N_S(T_2)$, this yields that $N_S(T_2)=Q$. Now it is easy to prove the remaining part of (ix).

Let C be an abelian subgroup of S of order at least 8. Since $|S:H|=2$, we have $|H\cap C|=4$, whence H contains $x^{2^{m-1}}$. Therefore, it follows that $C\subseteq C_S(x^{2^{m-1}})=H$. Thus (x) holds.

Now let N be a proper normal subgroup of S. If $N\subseteq H$, then either $N=H$ or $N\subseteq\langle x^2\rangle=S'$. Suppose that N is not containedin H. Then N must contains an element x^iy which is of order 2 or 4. If x^iy is an involution, then N contains all involutions in $S-H$ and so $\langle y, x^2y\rangle\subseteq N$, which implies that $N=\langle y, x^2y\rangle=\langle x^2, y\rangle$. If x^iy is of order 4, then N contains all elements of order 4 lying in $S-H$, and so $N=\langle xy, x^{-1}y\rangle=\langle x^2, xy\rangle$. Thus (xi) follows from (i).

Let D be a dihedral subgroup of S of order at least 8. Since $|S:H|=2$, it follows that $H\cap D$ is the unique maximal subgroup of D of index 2. Thus (xii) holds.

Since H is the unique maximal cyclic subgroup of S, it is characteristic in S. Hence if σ is an automorphism of S then the images of x and y under σ is of the form

$$x^\sigma=x^k,\quad k \text{ odd},$$

$$y^\sigma=x^iy,\quad i \text{ even}.$$

Conversely, a mapping $\sigma:S\rightarrow S$ defined as above is indeed an automorphism of S. Hence $|\mathrm{Aut}(S)|=2^m\cdot 2^m$. Thus (xiii) holds.

3. Proof of Theorem

In this section we prove our Theorem. Throughout this section G is a finite group with a quasi-dihedral Sylow 2-group S and the notation for S and the elements and subgroups of S are as in Lemma.

First of all, we will determine the focal subgroup $S\cap G'$. By Zassenhaus' theorem, $N_G(S)$ has a complement L to S. Since $N_G(S)/C_G(S)$ is a 2-group by Lemma (xiii), we have $L\subseteq C_G(S)$. Hence $N_G(S)=S\times L$, and we have

$S\cap N_G(S)'=S'$. Thus it follows from Grün's theorem [3] that

$$S\cap G'=\langle S', S\cap P' \mid P \text{ ranging over all Sylow 2-groups of } G\rangle.$$

If $P=S^g$ for some g in G, then $P'=S'^g=\langle x^2\rangle^g$, so $S\cap P'$ is cyclic of order at most 2^m. Moreover, if $|S\cap P'|\geq 8$ then $S\cap P'$ must be contained in H by Lemma (x), whence $S\cap P'\subseteq\langle x^2\rangle=S'$. And if $|S\cap P'|=4$ then we have either $S\cap P'=T_1\subseteq S'$ or $S\cap P'=\langle x^i y\rangle$ for some odd i by Lemma (ii). Finally, if $|S\cap P'|=2$ then we have either $S\cap P'=\langle z\rangle\subset S'$ or $S\cap P'=\langle x^i y\rangle$ for some even i. Hence we conclude that either $S\cap G'=S'$ or that

$$(*)\qquad S\cap G'=\langle x^2, x^{i_1}y, \dots, x^{i_r}y\rangle$$

for suitable integers i_k and Sylow 2-groups P_k such that $S\cap P_k'=\langle x^{i_k}y\rangle$, $1\leq k\leq r$. Furthermore, if (*) holds, then $S\cap G'=\langle x^2, xy\rangle$ if all i_k are odd, $S\cap G'=\langle x^2, y\rangle$ if all i_k are even, and $S\cap G'=\langle x^2, y, xy\rangle=S$ in the remaining case. Hence one of the following four cases occur:

$$S\cap G'=S', \qquad S\cap G'=\langle x^2, xy\rangle,$$
$$S\cap G'=\langle x^2, y\rangle, \quad S\cap G'=S.$$

Consider the case $S\cap G'=S$. Then G has no normal subgroups of index 2. In this case (*) holds and i_j is even and i_k is odd for suitable j, k, say $j=1$ and $k=2$. Since x^{i_1} and x^{i_2} are conjugate in S to y and xy, respectively, we may assume that there exist two Sylow 2-groups P_1 and P_2 of G such that

$$S\cap P_1'=\langle y\rangle \quad\text{and}\quad S\cap P_2'=\langle xy\rangle=T_2.$$

Thus $\langle y\rangle=\Omega_1(P_1')=Z(P_1)$ and $T_2=\Omega_2(P_2')$. Since $Z(P_1)$ is conjugate to $Z(S)=\langle z\rangle$ in G, the involution y is conjugate to z in G. Moreover, since $\Omega_2(P_2')$ is conjugate to $\Omega_2(S')=T_1$ in G, it follows that T_2 is conjugate to T_1 in G and that xy is conjugate $x^{2^{m-1}}$ in G. In particular, G has one conjugacy class of involutions and one of elements of order 4.

Moreover, by Lemma (viii) we have $|N_S(T):C_S(T)|=2$, so there exists $u\in N_S(T)$ such that

$$z^u=z,\quad y^u=zy,\quad (zy)^u=y.$$

On the other hand, $\langle y\rangle=Z(P_1)$, so the centralizer $C_G(y)$ contains both P_1 and $C_S(y)=T$. Let S_1 be a Sylow 2-group of $C_G(y)$ containing T. Then S_1 is a Sylow 2-group of G and $Z(S_1)=\langle y\rangle$. Moreover, as with S, we have $|N_{S_1}(T):C_{S_1}(T)|=2$, so there exists $v\in N_{S_1}(T)$ such that

$$y^v=y,\quad z^v=zy,\quad (zy)^v=z.$$

Thus if we set $w=uv$ then $w\in N_G(T)$ and w cyclically permutes the three involutions of T, whence w^3 is the least power of w contained in $C_G(T)$. Since $N_G(T)/C_G(T)$ is isomorphic to a subgroup of the symmetric group Σ_3, it follows that $N_G(T)/C_G(T) \cong \Sigma_3$ and $|N_G(T):C_G(T)|=6$.

On the other hand, the quaternion subgroup $Q=\langle T_1, T_2\rangle$ has exactly three subgroups of order 4. They are T_1, T_2 and $T_3=\langle x^{1+2^{m-1}}y\rangle$. Moreover, by Lemma (viii) and (ix), we have $|N_S(Q) : Q|=2$, $N_S(T_1)=S$ and $N_S(T_2)=Q$, so there exists $s\in N_S(Q)$ such that

$$T_1^s=T_1, \quad T_2^s=T_3, \quad T_3^s=T_2.$$

Since T_2 is the unique subgroup of P_2' of order 4, T_2 is normal inP_2. Hence $N_G(T_2)$ contains both P_2 and $N_S(T_2)=Q$. Let S_2 be a Sylow 2-group of $N_G(T_2)$ containing Q. Then S_2 is a Sylow 2-group of G and, as with$S$, we have $|N_{S_2}(Q) : Q|=2$, $N_{S_2}(T_2)=S_2$ and $N_{S_2}(T_1)=Q$. Thus there exists $t\in N_{S_2}(Q)$ such that

$$T_2^t=T_2, \quad T_3^t=T_1, \quad T_1^t=T_3,$$

Setting $r=st$, it follows that

$$T_1^r=T_3, \quad T_3^r=T_2, \quad T_2^r=T_1.$$

Hence $r\in N_G(Q)$ and r cyclically permutes T_1, T_2 and T_3, and so r^3 is the least power of r contained in $N_G(T_1)\cap N_G(T_2)$. Here we can show that $N_G(T_1)\cap N_G(T_2)=QC_G(Q)$. Note that, by the property of the quaternion group, if g is any element of $N_G(T_1)\cap N_G(T_2)$ then there exists an element h in Q such that $gh\in C_G(T_1)\cap C_G(T_2)=C_G(Q)$, which implies $g\in QC_G(Q)$. Therefore, we can conclude that $N_G(Q)/QC_G(Q)$ is isomorphic to Σ_3 and $|N_G(Q) : QC_G(Q)|=6$. Thus all parts of the assertion (1) in Theorem hold when $S\cap G'=S$.

Consider next the case $|S : S\cap G'|=2$. Then G has a normal subgroup K of index 2 such that $S\cap K=S\cap G'$ and $G/K\cong S/S\cap G'$, but G has no normal subgroups of index 4. Moreover, K has no normal subgroups of index 2. In fact, if K has a normal subgroup N of index 2, then there exists $g\in G$ such that $N^g\neq N$ and $N\cap N^g$ is normal in G. Since $G/N\cap N^g$ is of order 8, this group has a normal subgroup of index 4. Hence G has a normal subgroup of index 4, which is a contradiction.

Note that if $|S : S\cap G'|=2$ then we have either $S\cap G'=\langle x^2, y\rangle$ or $S\cap G'=\langle x^2, xy\rangle$. Consider first the case $S\cap G'=\langle x^2, y\rangle$. Then K has a dihedral Sylow 2-group $S\cap G'=\langle x^2, y\rangle$. Moreover, the preceding argument for (1) shows that y is conjugate to z in G, G has one conjugacy class of involutions and $|N_G(T):C_G(T)|=6$. But the element xy is not conjugate $x^{2^{m-1}}$ in G, for if $xy=(x^{2^{m-1}})^g$ with g in G, then $xy\in S\cap (S^g)'$, whence $S\cap G'=S$, contrary to our assumption. Hence G has two conjugacy classes of elements of order 4, and T_1 and T_2 are not conjugate in G. Therefore, we have $|N_G(Q):QC_G(Q)|=2$, otherwise this index would be 6 and $N_G(Q)$ would contain an element which cyclically permutes T_1, T_2 and T_3. But then T_1 and T_2 would be conjugate in G, which is not the case. Thus the assertion (2) in Theorem holds when $S\cap G'=\langle x^2, y\rangle$.

Now consider the case $S\cap G'=\langle x^2, xy\rangle$. Then K has a generalized quaternion Sylow 2-group $S\cap G'=\langle x^2, xy\rangle$. Moreover, the preceding argument for (1) shows that xy is conjugate to $x^{2^{m-1}}$ in G, G has one conjugacy class of elements of order 4, and $|N_G(Q):QC_G(Q)|=6$. On the other hand, the involution y is not conjugate to z in G, for if $y=z^g$ with g in G, then $y\in S\cap(S^g)'$, whence $S\cap G'=S$, contrary to our assumption. Thus G has two conjugacy class of involutions. And the only involution of S which is conjugate to z in G is z itself, so $Z(S)=\langle z\rangle$ is weakly closed in S with respect to G. In addition, we must have $|N_G(T):C_G(T)|=2$, otherwise this index would be 6 and $N_G(T)$ would contain an element which cyclically permutes the three involutions of T. But then y and z would be conjugate in G, which is not the case. Thus the assertion (3) in Theorem holds when $S\cap G'=\langle x^2, xy\rangle$.

Finally assume that $S\cap G'=S'$. Then there exists a normal subgroup K of G such that $S\cap K=S\cap G'=S'$ and $G/K\cong S/S'$. Hence S' is a Sylow 2-group of K and, being cyclic, K has a normal 2-complement L. Since L is characteristic in K, it follows that L is normal in G and G/L is a 2-group. Since $|L|$ is odd, we conclude that L is a normal 2-complement in G. The remaining parts of the assertion (4) in Theorem follow from this.

Thus we have completed the proof of Theorem.

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