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A CONTINUOUS OPERATOR VALUED REPRESENTATION ON A CERTAIN B^*-ALGEBRA

BY JAE CHUL RHO*

1. Introduction

Throughout this note S will denote a fixed nonempty compact set in the complexplane $\mathbf{C}$, Σ will be the Borel field of subsets of S, X and H are complex Banach space and complex Hilbert space respectively. Let $B(S, \Sigma)$ be the set of all uniform limit of finite linear combinations of characteristic functions of sets in Σ, then $B(S, \Sigma)$ forms a commutative B^*-algebra with the unit with respect to the supremum norm and the natural involution.

A fixed algebraic homomorphism $\psi : B(S, \Sigma) \rightarrow \mathcal{B}(X)$ will be called a continuous representation if $f_n \rightarrow f$ with respect to the supremum norm, then $\psi(f_n) \rightarrow \psi(f)$ with respect to the operator norm in $\mathcal{B}(X)$.

For a continuous representation ψ, if we put $A = \{\psi(f) : f \in B(S, \Sigma)\}$ then A forms a closed commutative subalgebra of $\mathcal{B}(X)$. If $X=H$, A is the commutative C^*-subalgebra with the unit $\psi(1)=I$, where the involution is determined by $\psi(f)^* = \psi(\bar{f})$.

If we put $\psi(\chi_\delta) = E(\delta)$ $(\delta \in \Sigma)$, then $E : \Sigma \rightarrow B(X)$ defines a spectral measure and any $\psi(f)$ can be represented by the integral form

$$\psi(f) = \int_S f(s) E(ds).$$

In addition if $E(\delta) = E(\delta)^*$ (self adjoint), then

$$\psi(f)^* = \int_S \bar{f}(s) E(ds), \quad \psi(f) \in A \subset B(H).$$

In this paper, we will determine the spectrum σ $(\psi(f))$, a relation between a scalar operator and an operator of multiplication by an independent variable. In the next, we introduce a complex measure by means of a certain continuous linear functional on $A \subset \mathcal{B}(H)$ and will be formulated an in tegral representation of elements of the dual space $B^*(S, \Sigma)$. Finally we consider conditions under which two continuous representations are unitarily equivalent.

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2. A relation betweenthe scalar operator and the operator of multiplication

2.1 LEMMA. *Let $\phi : B(S, \Sigma) \to \mathcal{B}(X)$ be a continuous representation. Then $\phi(f)=0$ if and only if $f=0$.*

Proof. We note that $B(S, \Sigma)$ is the set of all bounded Borel measurable functions (see 5, II, p. 891). Suppose that $f \neq 0$ (not identically 0 on S), the set $N(f)=\{s \in S : f(s) \neq 0\}$ is not empty and $N(f) \in \Sigma$. We put $N(f) = \delta$ and consider the integral $\int_\delta f(s) E(ds)$.

This is not a zero operator; For, if we define

$$\chi_\delta \frac{1}{f} = \begin{cases} \frac{1}{f}, & s \in \delta, \\ 0, & s \notin \delta, \end{cases}$$

then $\chi_\delta \frac{1}{f} \in B(S, \Sigma)$. Thus $\int_S \chi_\delta \frac{1}{f} E(ds)$ is defined. Hence

$$\int_\delta f E(ds) \int_S \chi_\delta \frac{1}{f} E(ds) = E(\delta),$$

and $E(\delta) \neq 0$ since $\sigma(E(\delta)) = \{0, 1\}$, Moreover,

$$\int_\delta f(s) E(ds) = \int_S (\chi_\delta f)(s) E(ds) = E(\delta)\phi(f), \quad \phi(f) \neq 0.$$

Therefore if an $f \in B(S, \Sigma)$ which is not identically 0 on S, then $\phi(f)$ can not be a zero operator, or equivalently $\phi(f)=0$ implies $f=0$ on S.
The converse is obvious.

2.2 COROLLARY. *A continuous representation $\phi : B(S, \Sigma) \to A$ is a bijection.*

In order to determine the spectrum $\sigma(\phi(f))$, we observe the following facts.

Any Σ-simple functions on S can be represented by

$$f = \sum_{i=1}^n \alpha_i \chi_{\delta_i}, \quad \bigcup_{i=1}^n \delta_i = S \text{ and } \delta_i \cap \delta_j = \square \ (i \neq j), \ \delta_i \in \Sigma \ (j=1, 2, \dots).$$

Since $\phi(f)E(\delta_i) = E(\delta_i)\phi(f)$ for $i=1, 2, \dots n$, each $\mathfrak{M}_i = E(\delta_i)X$ reduce the operator $\phi(f)$ and $\sum_{i=1}^n \oplus \mathfrak{M}_i = X$. Moreover, since $\phi(f)E(\delta_i) = \alpha_i E(\delta_i)$, $\phi(f)$ acts on $\mathfrak{M}_i$ as the multiplication by α_i, that is,

$$\phi(f)x = \alpha_i x \text{ for any } x \in \mathfrak{M}_i.$$

Hence we have

$$\sigma(\psi(f)\,|\,\mathfrak{M}_i) = \{\alpha_i\} \text{ for each } i.$$

It follows from the fact $\sigma(\psi(f)) = \bigcup_{i=1}^{n} \sigma(\psi(f)\,|\,\mathfrak{M}_i)$ that

$$\sigma(\psi(f)) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} = \text{range of } f.$$

In general, we have a following proposition.

2.3 PROPOSITION. $\sigma(\psi(f))$ =closure of $f(S)$ *for each* $f \in B(S, \Sigma)$.

Proof. It is easily be shown that if the closure of f does not vanish on S, then $\frac{1}{f} \in B(S, \Sigma)$ and so $\psi(f)^{-1} = \psi\left(\frac{1}{f}\right) \in A$. Thus A is a full subalgebra of $B(X)$, whence

(1) $\sigma_A(\psi(f)) = \sigma(\psi(f))$, where $\sigma_A(\psi(f))$ is the spectrum of $\psi(f)$ relative to A.

Furthermore, for any $f \in B(S, \Sigma)$, $\lambda - f$ is invertible if and only if λ does not contained in the closure of the range of f, in this case $(\lambda - f)^{-1} \in B(S, \Sigma)$. Thus $(\lambda I - \psi(f))^{-1} = \psi[(\lambda - f)^{-1}]$ exists if and only if λ does not contained in the closure of the range of f.
Therefore, we have

(2) $\sigma_A(\psi(f))$ =Closure of the range of f.

From (1) and (2), we have

$$\sigma(\psi(f)) = \text{Closure of } f(S).$$

If f is a continuous function on S, then $\sigma(\psi(f)) = f(S)$.

For a $T \in B(X)$, the spectrum $\sigma(T)$ is the nonempty compact subset in $\mathbf{C}$. Conversely, if S is any nonempty compact subset in $\mathbf{C}$, is there any operator $T \in B(X)$ such that $\sigma(T) = S$?. The answere to this question is following:

2.4 COROLLARY. *For any compact subset* $S \neq \square$ *in* $\mathbf{C}$

$$\sigma(\psi e) = S,$$

where $e : S \to S$ *is the function defined by* $e(s) = s$, $s \in S$.

For, since the function e is continuous on S, $e \in B(S, \Sigma)$. Thus $\sigma(\psi e) =$ range of $e = S$ by the proposition 2.3.

From proposition 2.3 and Corollary 2.4, we have the following immediate consequence.

2.5 COROLLARY. *For any continuous function* $f \in B(S, \Sigma)$.

$$\sigma(\phi(f)) = f(\sigma(\phi e))$$

2.6 THEOREM. *Let* $\phi : B(S, \Sigma) \to A$ *be a continuous representation and* $Q : B(S, \Sigma) \to B(S, \Sigma)$ *be the operator of multiplication by an independent variable in* S. *Then an operator* J *with the following diagram is commutative. i e.,*

$$\begin{array}{ccc} B(S,\Sigma) & \xrightarrow{\phi} & A \\ Q\big\downarrow & & \big\downarrow J, \\ B(S,\Sigma) & \xrightarrow{\phi} & A \end{array} \qquad J\phi = \phi Q$$

if and only if J *is the scalar operar* $\phi(e) = \int_S sE(ds)$.

If J *is a scalar operator* $\int_S sE(ds)$, *then* $J = \phi Q \phi^{-1}$.

Proof. Since

$$(\phi Q)(f) = \phi(Qf) = \int_S sf(s) E(ds) = \int_S (ef)(s) E(ds) = (\phi(e))(\phi(f))$$

for each $f \in B(S, \Sigma)$, $J\phi = \phi Q$ implies that $(J\phi)f = [(\phi e)\phi]f$ for any $f \in B(S, \Sigma)$, i. e., $J\phi = \phi(e)\phi$.

It follows from Corollary 2.2 that $J = \phi(e) = \int_S sE(ds)$.

Conversely, if $J = \phi(e)$, from the above calculation the diagram is commutative and $J\phi = \phi Q$. Thus $J = \phi Q \phi^{-1}$ holds.

In general, Let $Q : B(S, \Sigma) \to B(S, \Sigma)$ be the operator of the multiplication by $g \in B(S, \Sigma)$. Then the operator J_g with $J_g \phi = \phi Q_g$ if and only if $J_g = \phi(g) = \int_S g(s) E(ds)$. In this case $J_g = \phi Q_g \phi^{-1}$.

3. A complex measure induced by a linear functional

It is well known that a complex measure on the sigma algebra of subsets of a set is defined by means of the spectral measure as

$$\mu_{x,y}(\delta) = (E(\delta)x, y) \text{ for } \delta \in \Sigma \text{ and } x, y \in H.$$

Here we consider a complex measure induced by a continuous linear functional on $A = \{\phi(f) : f \in B(S, \Sigma)\} \subset \mathcal{B}(H)$, and will be formulated an integral representation of elements in the dual space $B^*(S, \Sigma)$ of $B(S, \Sigma)$.

3. 1 THEOREM. *Let $\psi : B(S,\Sigma)\to A\subset\mathcal{B}(H)$ be the continuous representation, and let ϕ be a continuous linear functional on $A=\{\psi(f) : f\in B(S,\Sigma)\}$ equipped with the strong operator topology. Then the formula*

$$\phi(E(\delta))=\mu_\phi(\delta),\quad \delta\in\Sigma$$

defines a complex measure on Σ. And each element $\psi'\phi$ in $B^*(S,\Sigma)$ can be represented by the form

$$(\psi'\phi)f=\int_S f(s)\mu_\phi(ds),$$

where ψ' is the dual of ψ.

Proof. It is obvious that $\mu_\phi(\Box)=0$.
We have to show $\mu_\phi:\Sigma\to\boldsymbol{C}$ is countably additive. For any disjoint family $\{\delta_i\}_{i=1}^\infty\subset\Sigma$, the sequence $\{E(\delta_i)\}_i$ is orthogonal projections and so $\{E(\delta_i)x\}_i$ is an orthogonal sequence of vectors in H for any $x\in H$. Therefore,

$$\|\sum_{i=1}^\infty E(\delta_i)x\|^2=\sum_{i=1}^\infty\|E(\delta_i)x\|^2=\|E(\bigcup_{i=1}^\infty\delta_i)x\|^2\leqslant\|x\|^2,$$

thus $\sum_{i=1}^\infty E(\delta_i)x$ is summable. It follows that $\sum_{i=1}^n E(\delta_i)x=E(\bigcup_{i=1}^n\delta_i)x$ converges to $\sum_{i=1}^\infty E(\delta_i)x=E(\bigcup_{i=1}^\infty\delta_i)x$. This means that $\sum_{i=1}^n E(\delta_i)$ converges to the operator $\sum_{i=1}^\infty E(\delta_i)$ with respect to the strong operator topology. Therefore, by the assumption on ϕ, $\phi\left(\sum_{i=1}^n E(\delta_i)\right)$ converges to $\phi\left(\sum_{i=1}^\infty E(\delta_i)\right)=\phi\left(E\left(\bigcup_{i=1}^\infty\delta_i\right)\right)$.
Since $\phi\left(\sum_{i=1}^n E(\delta_i)\right)=\sum_{i=1}^n\mu_\phi(\delta_i)$, we have $\sum_{i=1}^\infty\mu_\phi(\delta_i)=\mu_\phi\left(\bigcup_{i=1}^\infty\delta_i\right)$.

Now, since any $f\in B(S,\Sigma)$ is the uniform limit of some Σ-simple functons $\left\{\sum_{i=1}^n\alpha_i\chi_{\delta_i}\right\}_n$ and $\sum_{i=1}^n\alpha_iE(\delta_i)$ converges to $\psi(f)$ with respect to the uniform operator topology, whence $\left\{\sum_{i=1}^n\alpha_iE(\delta_i)\right\}_n$ converges strongly to $\psi(f)$. Therefore, $\phi\left(\sum_{i=1}^n\alpha_iE(\delta_i)\right)=\sum_{i=1}^n\alpha_i\mu_\phi(\delta_i)$ converges to $\phi(\psi(f))=\int_S f(s)\mu_\phi(ds)$. Furthermore, we may consider $\psi\in\mathcal{L}(B(S,\Sigma),A)$ (The set of all bounded linear operators), there corresponds a unique dual operator $\psi'\in\mathcal{L}(A^*,B^*(S,\Sigma))$ such that $\|\psi\|=\|\psi'\|$ and $\psi\circ\phi=\psi'\phi$. Hence we have

$$(\psi'\phi)f=\int_S f(s)\mu_\phi(ds),\quad f\in B(S,\Sigma).$$

It is evident that $\|\psi'\phi\|\leqslant\|\phi\|$ since $\|\psi(f)\|\leqslant\|f\|$.
This completes the proof.

In the Theorem 3. 1 we assumed ϕ is continuous on A equipped with the strong operator topology so that μ_ϕ is countably additive. If ϕ is continuous on A equipped with the uniform operator topology, then the same result holds as in the Theorem 3. 1 through simpler calculations. In this case we observe the followings:

Let A^* be the dual space of $A \subset B(H)$, then obviously A^* is closed with respect to the topology induced by the norm of a linear functional. We consider thestrong topology on A^*, namely that a sequence $\{\phi_n\}_n$ in A^* converges to ϕ if and only if

$$\phi_n(\psi(f)) \longrightarrow \phi(\psi(f)) \text{ for any } \psi(f) \in A.$$

And we denote the strong closure of A^* by A_S^*. Here we carefully distinguish the strong operator topology from the strong topology.

3. 2. PROPOSITION. *Let* $Y = \{\mu_\phi : \phi \in A_S^*\}$, *then* Y *is a Banach space with respect to the norm*

$$\|\mu_\phi\| = \sup\{|\mu_\phi(\delta)| : \delta \in \Sigma\}, \quad \|\mu_\phi\| \leqslant \|\phi\|.$$

Proof. It is easy to check that Y is a normed linear space. For the completeness, let $\{\mu_{\phi_n}\}_n$ be a cauchy sequence in Y, then

$$\|\mu_{\phi_n} - \mu_{\phi_m}\| \geq |\mu_{\phi_n}(\delta) - \mu_{\phi_m}(\delta)| \to 0 \text{ as } m, n \to \infty \text{ for any } \delta \in \Sigma.$$

Since each ϕ_n continuous on A equipped with the uniform operator topology, and any $\psi(f) \in A$ can be approximated by a sequence $\{\sum_{i=1}^{n} \alpha_i E(\delta_i)\}$,

$$|\phi_n(\psi(f)) - \phi_m(\psi(f))| \to 0 \quad \text{as} \quad m, n \to \infty \text{ for any } \psi(f) \in A.$$

Thus $\lim_n \phi_n(\psi(f))$ exists for each $\psi(f)$ in A. If we put

$$\lim_n \phi_n(\psi(f)) = \phi(\psi(f)),$$

then $\phi \in A_s^*$ and ϕ is a linear functional on A. Moreover, for any $\varepsilon > 0$ there exists an $N > 0$ such that

$$|\phi_n(\psi(f)) - \phi(\psi(f))| < \varepsilon \text{ for any } n \geq N.$$

Thus

$$|\phi(\psi(f))| < |\phi_N(\psi(f))| + \varepsilon \leq \|\phi_N\| \|\psi(f)\| + \varepsilon$$

so we have

$$|\phi(\psi(f))| \leq \|\phi_N\| \|\psi(f)\| \text{ for any } \psi(f) \in A.$$

It follows that ϕ is continuous on A equipped with the uniform operator topology, thus

$$\mu_\phi(\delta)=\phi(E(\delta)),\ \ \delta\in\Sigma$$

defines a complex measure and $\mu_\phi\in Y$.

Since $\sup_{\|\varphi(f)\|=1}|\phi(\psi(f))|=\|\phi\|$, obviously $\|\mu_\phi\|\leq\|\phi\|$.

We consider a set $\{\mu_\phi:\phi\in A^*\}$. This is a normed linear space with the same norm as stated above, and the map $\mu: A^*\to\{\mu_\phi:\phi\in A^*)$ defined by $\mu(\phi)=\mu_\phi$ is continuous since $\|\mu_\phi\|\leqslant\|\phi\|$.

It is not difficult to show the following

3.3 PROPOSITION. *A map $\mu: A_S{}^*\to Y$ defined by $\mu(\phi)=\mu_\phi$ may not be continuous, but it is linear, bijection and open (the inverse is continuous).*

EXAMPLE. Let H be a sepable Hilbert space. We shall obtain an explicit form of a linear functional on $\mathcal{B}(H)$ such that $\sum_{i=1}^\infty \phi(E(\delta_i))$ is summable.

Let $v=\left(\frac{\beta_1}{2^i},\ \frac{\beta_2}{(2^i)^2},\ldots\right)$, $|\beta_k|\leqslant1$ $(k=1,2,3,\ldots)$ for $i\in N$. And let P_1 be a projection operator to the first coordinate of the vector vT, $T\in B(H)$. If we put

$$\phi=P_1\circ v \quad\text{and}\quad \phi(E(\delta))=\mu_\phi(\delta)\ \ (\delta\in\Sigma)$$

then μ_ϕ is a complex measure such that $\sum_{i=1}^\infty \mu_\phi(\delta_i)$ is summable for any disjoint family $\{\delta_i\}_i$ in Σ.

For, since each operator on H can be represented by a matrix (a_{ik}) with $\sum_{i=1}^\infty|a_{ik}|^2<\infty$ $(k=1,2,\ldots)$. (We note that if the operator is the form $\psi(f)$, then each a_{ik} is a function of f.)

Therefore

$$\phi(T)=\sum_{j=1}^\infty\frac{\beta_j}{(2^i)^j}a_{j1},\quad |\phi(T)|\leqslant\sum_{j=1}^\infty\frac{1}{(2^i)^j}|a_{j1}|<\infty$$

by the Schwartz inequality. Moreover since each $E(\delta_i)$ is a projection operator on H, some part of the diagonal elements are equal to 1 and remaining elements are zero. Therefore,

$$|\phi(E(\delta_i))|\leqslant\frac{1}{2^i}\ \text{and}\ \sum_{i=1}^\infty|\phi(E(\delta_i))|\leq1$$

Thus $\sum_{i=1}^\infty \mu_\phi(\delta_i)$ is summable.

In this example,

$$\phi(\psi(f))=\int_S f(s)\mu_\phi(ds)=\sum_{j=1}^{\infty}\frac{\beta_j}{(2^i)^j}a_{j1}(f).$$

We leave, however, the following questions:

(1) If H is a separable Hilbert space, is there another kind of an explicit form of a linear functional on $\mathcal{B}(H)$ other than the stated above such that $\sum_{i=1}^{\infty}\phi(E(\delta_i))$ is summable?

(2) Let H be a separable Hilbert space. For any linear functional on $B(H)$, is there any explicit form such that $\sum_{i=1}^{\infty}\phi(E(\delta_i))$ is summable?

4. A unitary equivalence of two continuous representations

Let $\psi, \varphi : B(S, \Sigma) \to \mathcal{B}(H)$ be two continuous representations. We put $\psi(\chi_\delta)=E(\delta)$ and $\varphi(\chi_\delta)=F(\delta)$ for $\delta\in\Sigma$. Then it is easy to show that $E(\delta)$ and $F(\delta)$ are unitarily equivalent for any $\delta\in\Sigma$ if and only if $\psi(f)$ and $\varphi(f)$ are unitarily equivalent for any $f\in B(S, \Sigma)$.

Now, we will find conditions under which two representations are unitarily equivalent.

4.1 DEFINITION. Two continuous representations are said to be unitarily equivalent with respect to $B(S, \Sigma)$ if there exists a unitary operator U such that $U^*\psi(f)U=\varphi(f)$ for all $f\in B(S, \Sigma)$. We denote it by $U^*\psi U=\varphi$ w. r. t. $B(S, \Sigma)$.

4.2 DEFINITION. A representation (not necessarily continuous) $\psi : B(S, \Sigma) \to \mathcal{B}(X)$ is called cyclic if there exists a vector $x\in X$ such that the set $\{\psi(f)x : f\in B(S, \Sigma)\}$ is dense in X. In this case x is said to be a cyclic vector.

If $\{\psi(f)x : f\in B(S, \Sigma)\}=X$, ψ is called a strictly cyclic representation and x is said to be a strictly cyclic vector.

4.3 PROPOSITION. *Let $\psi : B(S, \Sigma)\to A\subset\mathcal{B}(X)$ be a (not necessarily continuous) strictly cyclic representation. Then $A=\{\psi(f) : f\in B(S, \Sigma)\}$ is the maximal abelian subset of $\mathcal{B}(X)$.*

Proof. Let x be a strictly cyclic vector, then $Tx\in X$ for any $T\in B(X)$. Hence there exists an $f\in B(S, \Sigma)$ such that $Tx=\psi(f)x$. If $T\psi(g)=\psi(g)T$ for any $g\in B(S, \Sigma)$, then

$$T\psi(g)x=\psi(g)Tx=\psi(g)\psi(f)x=\psi(f)\psi(g)x.$$

Thus we have $T=\psi(f)$, therefore A is the maximal Abelian.

Now, we shall show that a condition for which two continuous representations are unitarily equivalent.

We consider a subset $B_0(S, \Sigma)=\{f\in B(S, \Sigma)$: closure $f(S)\not\ni 0\}$ of $B(S, \Sigma)$ and put $A_0=\{\psi(f)\in\mathcal{B}(H) : f\in B_0(S, \Sigma)\}$.

4.4. THEOREM. *Let $\psi : B(S, \Sigma)\to\mathcal{B}(H)$ be a (continuous) representation such that there exists a vector x with A_0x is dense in H. And let φ be another cyclic representation with a cyclic vector y. If $(E(\delta)x, x)=(F(\delta)y, y)$ for any $\delta\in\Sigma$ then ψ and φ are unitarily equivalent with respect to $B_0(S, \Sigma)$, where $F(\delta)=\varphi(\chi_\delta)$, $\delta\in\Sigma$.*

Proof. Since $A_0\subset A$, obviously ψ is cyclic and x is a cyclic vector. Moreover $(E(\delta)x, x)=(F(\delta)y, y)$ for each $\delta\in\Sigma$ implies $(\psi(f)x, x)=(\varphi(f)y, y)$ for any $f\in B(S, \Sigma)$.

We define an operator U such a way that if

$$U\psi(f)x=\psi(f)y, \quad f\in B(S, \Sigma)$$

then $Ux=y$ and U is densely defined linear operator with the range is also dense in H. Moreover, since $\psi(|f|^2)=\psi(\bar{f})\psi(f)=\psi(f)^*\psi(f)$, we have

(1) $(U\psi(f)x,\ U\psi(f)x)=(\varphi(f)y, \varphi(f)y)=(\psi(f)x, \psi(f)x)$.

Thus U is bounded lnear on a dense subset of H, whence U is defined on H. Here we denote the extension $\bar{U}$ of U, defined by $\bar{U}(\lim_n x_n)$, $Ux_n=\bar{U}x_n$ for each n, by the same symbol U.

From the assumption, for any $z\in H$ there exists a sequence $\{f_n\}_n$ in $B_0(S, \Sigma)$ such that $\psi(f_n)x\to z$. It follows from (1) that

(2) $(Uz, Uz)=(z, z)$ for any $z\in H$.

And for any $u\in H$ there exists a sequence $\{g_n\}_n$ in $B(S, \Sigma)$ such that $\varphi(g_n)y\to v$. Thus $U\psi(g_n)x=\psi(g_n)y\to v$. That is, $Uu=v$, where $u=\lim_{n\to\infty}\psi(g_n)\ x\in H$.

Hence U is a surjection. This fact together with (2) implies that U is a unitary operator, namely $(Ux, Uy)=(x, y)$ for any x and y in H. Thus $U^*U=UU^*=I$.

Since $U\psi(f)x=\psi(f)y$ for any $f\in B(S, \Sigma)$, we have

$$[\psi(f)-U^*\psi(f)U]x=0, \quad f\in B(S, \Sigma).$$

And since $I=\psi(1)=\psi\left(\frac{1}{f}\right)\psi(f)$ for any $f\in B_0(B, \Sigma)$, it follows that $\psi(f)=U^*\varphi(f)U$ on the dense subset of H.

From this and the fact that $\psi(f)-U^*\varphi(f)U$ is continuous on H, we have

$\psi(f)=U^*\varphi(f)U$ on H for any $f\in B_0(S,\Sigma)$, that is,

$$\psi=U^*\varphi U \text{ w.r.t. } B_0(S,\Sigma).$$

We have proved the proposition.

In the above discussions, we may consider the cyclic vector y belongs to another Hilbert space K, $H\neq K$, and we define $V: H\to K$ by

$$V\psi(f)x=\varphi(f)y \text{ for each } f\in B(S,\Sigma).$$

Then similar arguments as above, we have $(Vu, Vv)_K=(u,v)_H$ for any u and v in H. Thus we have the following result:

4.4 PROPOSITION. *Let* $\psi : B(S,\Sigma)\to\mathcal{B}(H)$ *be a (continuous)* representation such that there exists a vector x with A_0x dense in H. And let $\varphi : B(S,\Sigma)\to\mathcal{B}(K)$ be a continuous cyclic representation with a cyclic vector y. If $(E(\delta)x,x)=(F(\delta)y,y)$ for any $\delta\in\Sigma$, then there exists an isometric operator $V: H\to K$ such that $\psi=V^*\varphi V$ w.r.t. $B_0(S,\Sigma)$.

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Sogang University