

CENTRAL PERFECT RINGS

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1. Introduction

Steinitz rings are like division rings in that every linearly independent subset of a free module can be extended to a basis by adjoining elements of a given basis [4]. The Wedderburn structure theorem classifies semisimple Artinian rings as the finite direct sums of complete matrix rings over division rings.

In this paper we study those rings which are finite direct sums of complete matrix rings over Steinitz rings. For reasons explained below we call these rings *central perfect rings*. A central perfect ring is to a Steinitz ring what a semisimple Artinian ring is to a division ring. In the (generalized) Maschke theorem [5] conditions are given for a group ring $R[G]$ to be a semisimple Artinian ring. In this paper we are among other things concerned with giving conditions for a group ring $R[G]$ to be central perfect. In dealing with this problem we have to say quite a few things about central perfect rings in general. Furthermore, the solutions we have given require the introduction of several tools we have found interesting in their own right. Accordingly, we have discussed these tools in more detail than absolutely necessary to consider only the problem of classification of central perfect group rings.

Since being 'perfect' is a one-sided notion, we shall mean 'right perfect' whenever the word 'perfect' is used. For left perfect rings all arguments have obvious adaptations.

2. Central perfect rings and decompositions

R will always denote a ring with an identity and JR denotes the Jacobson radical of R .

An additive idempotent mapping $f: R \rightarrow R$ such that $f(xf(y)) = f(f(x)y) = f(x)f(y)$ for all $x, y \in R$ will be called a *decomposition* of R . A decomposition $f: R \rightarrow R$ will be called a *normal decomposition* of R provided:

- (1) $\ker f \subseteq JR$; (2) $f(R)$ is a central perfect ring.

Normal decompositions will be an important tool in our investigation of central perfect rings. A decomposition satisfying condition (1) above will be called a *radical decomposition*.

PROPOSITION 1. *Central perfect rings are perfect.*

Proof: Suppose A is a T -nilpotent ring. König's lemma [3] implies that if $\{F_i\}$ is a sequence of finite subsets of A , then there is an integer n such that $F_n \dots F_1 = 0$. It follows that A_n is also T -nilpotent. Furthermore it is clear that finite direct sums of T -nilpotent rings and subrings of T -nilpotent rings are T -nilpotent.

Since $J(R_1 \oplus \dots \oplus R_n) = JR_1 \oplus \dots \oplus JR_n$ and $J(R_m) = (JR)_m$ (here R_m is the complete ring of $m \times m$ matrices over R), it follows that if S_1, \dots, S_n is a collection of Steinitz rings, then $(S_1)_{m_1} \oplus \dots \oplus (S_n)_{m_n}$ is a perfect ring.

Many examples of perfect rings and perfect group rings which are not central perfect will be given. Consequently proposition 1 implies that the problem of characterizing central perfect rings reduces to determining those perfect rings that are central perfect.

Let R denote a perfect ring. Then R/JR is a semisimple Artinian and thus $R/JR = (D_1)_{m_1} \oplus \dots \oplus (D_n)_{m_n}$, where D_j is a division ring. In particular, R/JR contains a complete set of primitive orthogonal idempotents $\bar{e}_1, \dots, \bar{e}_t$ ($t = m_1 + \dots + m_n$) and a complete orthogonal set of centrally primitive idempotents $\bar{f}_1, \dots, \bar{f}_n$ where

$$\bar{f}_1 = \bar{e}_1 + \dots + \bar{e}_{m_1}, \dots, \bar{f}_n = \bar{e}_{t-m_n+1} + \dots + \bar{e}_t,$$

without loss of generality.

Using standard techniques (e. g. [6]), we can lift these idempotents since JR is a nil ideal. We take e_1, \dots, e_t to be a complete set of primitive orthogonal idempotents in R , where $1 = e_1 + \dots + e_t$, and $e_i + JR = \bar{e}_i$. Letting $f_1 = e_1 + \dots + e_{m_1}, \dots, f_n = e_{t-m_n+1} + \dots + e_t$, we observe that $1 = f_1 + \dots + f_n$, $f_i + JR = \bar{f}_i$ and f_1, \dots, f_n is a complete orthogonal set of idempotents. Later examples will show that the f_i 's need not be central even though the \bar{f}_i 's were central.

Define $f : R \rightarrow R$ by $f(r) = r^* = \sum_{i=1}^n f_i r f_i$. We will refer to f as a *standard normal decomposition*.

THEOREM 1: *If R is a perfect ring, then any standard normal decomposition is a normal decomposition of R .*

Proof: Clearly, $(x+y)^* = \sum f_i (x+y) f_i = x^* + y^*$, $(x^*)^* = \sum f_i (\sum f_j x f_j) f_i = \sum f_i^2 x f_i^2 = \sum f_i x f_i^2 = \sum f_i x f_i = x^*$. Also, $(x^* y)^* = \sum f_i (x^* y) f_i = \sum f_i (\sum f_j x f_j y) f_i = \sum f_i^2 x f_j y f_i = \sum f_i x f_j^2 y f_j = \sum (f_i x f_j) (f_j y f_j) = (\sum f_i x f_j) (\sum f_j y f_j) = x^* y^*$.

Similarly $(xy^*)^* = x^* y^*$. Thus the mapping is a decomposition.

Let $R^* = f(R)$ and $M = \ker f$. We must show that $M \subseteq JR$ and that R^* is central perfect.

Since $1^* = 1$, M is a unitary R^* -bimodule. Furthermore, since $r + JR = \bar{r} = \sum \bar{f}_i r f_i$ and since $r^* = JR = \sum \overline{f_i r f_i}$, $r - r^* \in JR$ and thus $M \subseteq JR$, i. e., f is a radical decomposition. Now, $f_i^* = \sum f_j f_i f_j = f_i$, whence $f_i \in R^*$ and $f_i r^* = f_i (\sum f_j r f_j) = f_i r f_i$, $r^* f_i = f_i r f_i$, i. e., f_i is in the center of R^* .

We have $e_i^* = \sum f_j e_i f_j$, and by the orthogonality of the e_i , $e_1^* = f_1 e_1 f_1 = e_1$, so that the primitive idempotents are also in R^* .

Since R perfect it is semi-perfect and thus if e is a primitive idempotent, eRe is a local ring. Thus the rings $e_i Re_i$ are local. In fact it is easily seen that $e_i JR e_i = e_i Re_i \cap JR$, $e_i JR e_i = J(e_i Re_i)$, and thus $J(e_i Re_i)$ is T -nilpotent, i. e., $e_i Re_i$ is a Steinitz ring, since Steinitz rings are perfect local rings [4]. Now, $f_i R f_i$ has radical $J(f_i R f_i) = f_i JR f_i = f_i R f_i \cap JR$ by essentially the same argument as that for $e_i Re_i$, and since $f_i R f_i / J(f_i R f_i) = \bar{f}_i (R/JR) \bar{f}_i$ is prime, it follows that $f_i R f_i$ is the complete ring of $m_j \times m_j$ matrices over one of the Steinitz rings $S_j = e_{n(j)} Re_{n(j)}$, where $e_{n(j)}$ is one of the local idempotents which occur in the expression for f_j .

Thus, since $R^* = \sum f_i R f_i$ is a ring direct sum of these rings, it follows that R^* is a central perfect ring and that f is a normal decomposition as asserted.

COROLLARY 1; *A perfect ring R is central perfect if and only if the idempotents $e_1, \dots, e_t, f_1, \dots, f_n$ can be chosen in such a way that the idempotents f_1, \dots, f_n are themselves in the center of R .*

Proof: If the f_i are in the center of R , then $r(f_1 + \dots + f_n) = r = \sum f_i r f_i = r^*$ and $M = 0$, i. e., $f(r) = r = r^*$ is the identity map, whence $f(R) = R^* = R$ is central perfect. On the other hand, if R is central perfect, then $R = (S_1)_{m_1} \oplus \dots \oplus (S_n)_{m_n}$ and if we select the e_i to be the appropriate matrices with one 1 on the diagonal and 0's elsewhere, then f_1, \dots, f_n will be the identities for $(S_1)_{m_1}, \dots, (S_n)_{m_n}$ respectively, whence they themselves are in the center of R .

The terminology "central perfect" was arrived at from the observations that central perfect rings are perfect and that the complete orthogonal set of centrally primitive idempotents $\bar{f}_1, \dots, \bar{f}_n$ of R/JR can be lifted to a complete orthogonal set of centrally primitive idempotents f_1, \dots, f_n .

Given decompositions f and g of R , we define an equivalence relation $f \equiv g$ provided there is an automorphism α of R such that $f = \alpha \cdot g \cdot \alpha^{-1}$. In this case the decompositions f and g are conjugate. Notice that $\alpha \cdot g \cdot \alpha^{-1}$ is a decomposition of R whenever g is a decomposition of R and α is an automorphism.

Since automorphisms leave the Jacobson radical invariant, it follows that $\alpha \cdot g \cdot \alpha^{-1}$ is a radical decomposition whenever g is a radical decomposition. If R is perfect and if g is a normal decomposition, then $\alpha \cdot g \cdot \alpha^{-1}$ is also a normal decomposition.

Now suppose that g is a standard normal decomposition with associated primitive (local) idempotents e_1, \dots, e_t and associated central idempotents f_1, \dots, f_n . Thus, $\alpha \cdot g \cdot \alpha^{-1}(r) = \alpha^{-1}(\sum f_i \alpha(r) f_i) = \sum \alpha^{-1}(f_i) r \alpha^{-1}(f_i)$, $\alpha^{-1}(e_1) + \dots + \alpha^{-1}(f_1) + \dots + \alpha^{-1}(f_n) = \alpha^{-1}(e_{t-m_n+1}) + \dots + \alpha^{-1}(e_t)$, that is, $\alpha \cdot g \cdot \alpha^{-1}$ is a standard normal decomposition, with associated primitive (local) idempotents $\alpha^{-1}(e_1), \dots, \alpha^{-1}(e_t)$ and associated central idempotents $\alpha^{-1}(f_1), \dots, \alpha^{-1}(f_n)$.

THEOREM 2: *Let f and g be standard normal decompositions of the perfect ring R . Then f and g are conjugate decompositions.*

Proof: Suppose f has associated primitive (local) idempotents e_1, \dots, e_t and g has associated primitive idempotents a_1, \dots, a_l .

By Azumaya's theorem $t=l$ and there exists a unit v of R and a permutation P of the numbers $1, \dots, t$ such that $ve_i = a_{p(i)}v$.

Let $\alpha : R \rightarrow R$ be the inner automorphism $\alpha(r) = vr v^{-1}$.

Then $\alpha^{-1}(a_{p(i)}) = e_i$, $\alpha(e_i) = a_{p(i)}$ and since $g(a_{p(i)}) = a_{p(i)}$ (as in theorem 1), $(\alpha^{-1} \cdot g \cdot \alpha)(e_i) = e_i$. Thus $\alpha \cdot g \cdot \alpha^{-1}$ and f are normal decompositions which map the elements e_i to the elements e_i . Since the central idempotents associated with standard normal decompositions are minimal among the central idempotents which can be constructed as sums of the local idempotents associated with these standard normal decompositions, it follows that if two standard normal decompositions have the same associated local idempotents, then they have the same associated central idempotents. But then it follows that the mappings are themselves identical. Thus, in our case $f = \alpha \cdot g \cdot \alpha^{-1}$, i. e., $f \equiv g$.

Now suppose that $g : R \rightarrow R$ is a normal decomposition. Then $g(R) = A$ is a central perfect ring, say $A = (S_1)_{m_1} \oplus \dots \oplus (S_n)_{m_n}$. If f_i is the idempotent which acts as the identity on $(S_i)_{m_i}$ and which annihilates $(S_j)_{m_j}$ if $j \neq i$, then $1 = f_1 + \dots + f_n$ and since $(S_j)_{m_j}$ contains no central idempotents other than 0 or 1, f_1, \dots, f_n is a complete orthogonal set of centrally primitive idempotents. Also, $f_1 = e_1 + \dots + e_{m_1}$, \dots , $f_n = e_{t-m_n+1} + \dots + e_t$, $t = m_1 + \dots + m_n$, where the e_i 's correspond to the appropriate matrices with a single 1 on the main diagonal. Then, since Steinitz rings contain no idempotents other than 0 or 1, e_1, \dots, e_t is a complete orthogonal set of primitive idempotents.

Since $R = A \oplus \ker g$, and since $\ker g \subseteq JR$, it follows that the set $e_1 + JR, \dots, e_t + JR$ is a complete orthogonal set of primitive idempotents in the semi-

simple Artinian ring R/JR . Thus the idempotents f_1, \dots, f_n can be used in constructing a standard normal decomposition $f: R \rightarrow R$ as above. If we let $R^* = f(R)$, then since the f_i are central idempotents of A , it follows that $g(R) \subseteq R^* = f(R)$. Using this observation along with theorem 2 we have the following result.

COROLLARY 1: *Suppose R is a perfect ring and suppose $f: R \rightarrow R$ is a standard normal decomposition of R . If $g: R \rightarrow R$ is any normal decomposition of R , then there is an automorphism α of R such that $(\alpha \cdot g \cdot \alpha^{-1})(R) \subseteq f(R)$.*

Standard normal decompositions of perfect rings therefore give essentially unique best possible decompositions of perfect rings as direct sums of central perfect rings and remainders, the kernels of the standard normal decompositions.

We close this section with a proposition which is a counterexample, makes use of the ideas developed in theorems 1 and 2 and which uses a construction variants of which will be used below. Furthermore we show that not all normal decompositions are standard normal decomposition.

PROPOSITION 2: *Not every perfect ring is central perfect.*

Proof: Suppose R is any Steinitz ring. Let $m > 1$ and let $T_m(R)$ be the ring of lower triangular matrices with coefficients in R . Thus, if $X \in T_m(R)$, then $X_{ij} = 0$ if $j > i$. It follows that $JT_m(R)$ consists of all matrices X with $X_{ii} \in JR$ for all i . Now, $T_m(R)/JT_m(R) = R/JR \oplus \dots \oplus R/JR$, which is the direct sum of division rings.

Also, since the matrices with 0's on the diagonal form a nilpotent ideal and since JR is T -nilpotent, then by König's lemma $JT_m(R)$ is T -nilpotent, i. e., $T_m(R)$ is perfect.

The set E_{11}, \dots, E_{mm} , where E_{ii} denotes a matrix with a 1 in the (i, i) position and 0's elsewhere yields a complete orthogonal set of primitive and centrally primitive idempotents in $T_m(R)/JT_m(R)$, and hence we may use these to construct a standard normal decomposition $f: T_m(R) \rightarrow T_m(R)$ with $f(X) = \sum E_{ii} X E_{ii}$, i. e., $f(X)$ is the diagonal part of X . Since $\ker f$ consists of all matrices with 0 on the diagonal, $\ker f \neq 0$, whence $T_m(R)$ is not central perfect.

Notice that $X = f(X) + (X - f(X))$ writes X as the sum of its diagonal part and its off-diagonal part in this standard normal decomposition.

EXAMPLE 1. Let F be a field and let R be the ring of 3×3 lower triangular matrices with constant diagonals and coefficients in F . Then R is a Steinitz ring and JR consists of the matrices with 0's on the diagonal.

Thus, $(JR)^2 \neq 0$, $(JR)^3 = 0$. Let $S = R \oplus JR$, with $(r, s)(r', s') = (rr', sr' + rs')$. Then S is a Steinitz ring with $JS = JR \oplus JR$. Define $f : S \rightarrow S$ by $f(r, s) = (r, 0)$ and $g : S \rightarrow S$ by $g(r, j) = (r - j, 0)$. Then f and g are normal decompositions with $f(S) = g(S) = R$. Now $(\ker f)^2 = 0$ and $(\ker g)^2 \neq 0$. Thus f and g are not conjugate and since S is a Steinitz ring, they are not the standard normal decomposition since only the identity map is a standard normal decomposition. We note that $f \cdot g = f$ and $g \cdot f = g$.

3. Semi-direct sums

In the previous section the notion of decomposition was used to obtain a structure theorem for perfect rings involving central perfect rings. In this section and the following section we study decompositions in a more general setting both to obtain a better idea of the nature of a decomposition and to derive a stock of results which will be useful later on. In the following section we will also give a variety of examples, counterexamples and constructions involving decompositions and perfect rings.

In this section we demonstrate the (usual) equivalence between the (interior) notion of a decomposition and the (exterior) notion of a direct sum. The direct sums we deal with are not ring direct sums although the end result is a ring. These are the *semi-direct sums* discussed in this section.

Let R be a ring and let M be an R -bimodule, i. e., M is a left and right R -module such that for all $r_1, r_2 \in R$ and $m \in M$, $(r_1 m)r_2 = r_1(mr_2)$. If $S = R \oplus M$ is the direct sum of the R -bimodules R and M , then S is itself an R -bimodule. A *multiplication* ϕ on M is quite simply a mapping $\phi : M^2 \rightarrow S$ such that if we define the product on S by $(r_1 + m_1)(r_2 + m_2) = r_1 r_2 + m_1 r_2 + r_1 m_2 + \phi(m_1, m_2)$, then S becomes a ring. We denote this ring by $R \oplus_\phi M$, and we shall refer to $R \oplus_\phi M$ as a semi-direct sum of R and the bimodule M .

LEMMA 1: *Let M be an R -bimodule and let $S = R \oplus M$. Then $\phi : M^2 \rightarrow S$ is a multiplication if and only if ϕ is a bi-additive mapping which satisfies the following additional properties:*

- (1) $\phi(rm_1, m_2) = r\phi(m_1, m_2)$;
- (2) $\phi(m_1, m_2r) = \phi(m_1, m_2)r$;
- (3) $\phi(m_1, rm_2) = \phi(m_1r, m_2)$;
- (4) *if $\phi(m_1, m_2) = r + m$, $\phi(m_2, m_3) = r' + m'$, then*
 $rm_3 + \phi(m, m_3) = m_1r' + \phi(m_1, m')$.

Proof: Suppose that ϕ is a multiplication, i. e., $S = R \oplus_\phi M$ is a ring. Then (1), (2), (3) and (4) are consequences of the associative law. Thus

these properties are necessary.

On the other hand, if $\phi : M^2 \rightarrow S$ is a bi-additive map satisfying properties (1), (2), (3) and (4), then it is a straight forward matter to show that $R \oplus_{\phi} M$ is indeed a ring.

LEMMA 2: Suppose that $S = R \oplus_{\phi} M$ is a semi-direct sum of R and M . Define $f : S \rightarrow S$ by $f(r+m) = r$. Then f is a decomposition of S .

Proof: Since $S = R + M$, it follows that f is an additive idempotent function on S . Also if $x = r_1 + m_1$, $y = r_2 + m_2$, then

$$f(f(x)y) = f(r_1 r_2 + r_1 m_2) = r_1 r_2 = f(x)f(y) \text{ and } f(xf(y)) = f(x)f(y).$$

LEMMA 3: Suppose that $f : S \rightarrow S$ is a decomposition. Let $R = f(S)$, $M = \ker f$ and let $\phi : M^2 \rightarrow S$ be given by $\phi(m_1, m_2) = m_1 m_2$. Then $S = R \oplus_{\phi} M$, and the decomposition given in lemma 2 is f itself.

Proof: Since f is additive and idempotent we may write $S = R \oplus M$, where the direct sum is of groups. Since $f(f(x)y) = f(x)f(y)$, it follows that $f(S) = R$ is a subring of S . Furthermore since $f(f(x)m) = f(x)f(m) = 0$, $f(mf(y)) = f(m)f(y) = 0$, and since $(m_1 f(x))m_2 = m_1(f(x)m_2)$, it follows that M is an R -bimodule. Since S is a ring, ϕ is a multiplication and $S = R \oplus_{\phi} M$ is a semi-direct sum. Since $f(x) = f(f(x) + (x - f(x)))$, and $f(x - f(x)) = 0$, it follows that $f(r+m) = r$, i.e., f is the decomposition associated with the semi-direct sum $S = R \oplus_{\phi} M$ as in lemma 2.

LEMMA 4: If $S = R \oplus_{\phi} M$ is a semi-direct sum, and if f is the associated decomposition, then $R = f(S)$, $M = \ker f$, and $R \oplus_{\phi} M$ is the semi-direct sum associated with f as in lemma 2.

Proof: Entirely straightforward.

Now, let C_1 be the category whose objects are pairs (S, f) where $f : S \rightarrow S$ is a decomposition. Given pairs (S, f) and (S', f') a morphism $\alpha : (S, f) \rightarrow (S', f')$ is a ring homomorphism $\alpha : S \rightarrow S'$ such that $\alpha \cdot f' = f' \alpha$. Quite clearly, the composition of morphisms is a morphism and since identity maps are morphisms, it follows readily that C_1 is indeed a category.

If $R = f(S)$, $M = \ker f$, $R' = f'(S')$, $M' = \ker f'$, then

$$\alpha(R) = \alpha(f(S)) = f'(\alpha(S)) \subseteq R', \text{ and } f'(\alpha(r)) = \alpha(f(M)) = 0,$$

whence $\alpha(M) \subseteq M'$, i.e., $\alpha : M \rightarrow M'$.

Similarly, let C_2 be the category whose objects are triples (R, M, ϕ) , where R is a ring, M is an R -bimodule and $\phi : M^2 \rightarrow R \oplus M$ is a multiplication. Given triples (R, M, ϕ) and (R', M', ϕ') , a morphism $(R, M, \phi) \rightarrow (R', M', \phi')$ is a pair (β, β^*) where $\beta : R \rightarrow R'$ is a ring homomorphism and $\beta^*(rm) = \beta(r)\beta^*(m)$, $\beta^*(mr) = \beta^*(m)\beta(r)$ and if $\phi(m_1, m_2) = r + m$, then

$$\phi'(\beta^*(m_1), \beta^*(m_2)) = \beta(r) + \beta^*(m).$$

Then again it can be easily seen that the composition of morphisms $(\beta, \beta^*) \cdot (\gamma, \gamma^*) = (\beta \cdot \gamma, \beta^* \cdot \gamma^*)$ is a morphism and that $(1_R, 1_M)$ is the identity morphism. It is then a simple matter to show that C_2 is indeed a category.

LEMMA 5: *If $T: C_1 \rightarrow C_2$ is defined by $T(S, f) = (f(S), \text{Ker } f, \phi)$, where $\phi: \text{Ker } f^2 \rightarrow S$ is the restriction of ordinary multiplication, and if $T(\alpha) = (\alpha|f(S), \alpha|\text{Ker } f)$ for morphisms $\alpha: (S, f) \rightarrow (S', f')$, then T is a covariant functor.*

Proof: Straightforward.

LEMMA 6: *If $U: C_2 \rightarrow C_1$ is defined by $U(R, M, \phi) = (R \oplus_\phi M, f)$, where f is the standard projection of $R \oplus_\phi M$ onto R , and $U(\beta, \beta^*) = \alpha$, where $\alpha(r+m) = \beta(r) + \beta^*(m)$ for morphisms $(\beta, \beta^*): (R, M, \phi) \rightarrow (R', M', \phi')$, then U is a covariant functor.*

Proof: We have

$\alpha(r_1+m_1)(r_2+m_2) = \beta(r_1)\beta(r_2) + \beta(r_1)\beta^*(m_2) + \beta^*(m_1) + \beta(r_2) + \beta(r) + \beta^*(m)$ where $\phi(m_1, m_2) = r+m$. Hence from the properties of β^* and β , it follows directly that α is in fact a ring homomorphism. Also, $\alpha f(r+m) = \alpha(r)$, and $f'\alpha(r+m) = f'(\beta(r) + \beta^*(m)) = f'(\beta(r)) = \beta(r) = \alpha(r)$, i. e., $f' \cdot \alpha = \alpha \cdot f'$, whence α is a morphism in C_1 . It is now easy to show that U is in fact a covariant functor.

THEOREM 3: *with the categories C_1 and C_2 as defined above, and with the functors $T: C_1 \rightarrow C_2$ and $U: C_2 \rightarrow C_1$ as given, we have $T \cdot U = I(C_1)$ and $U \cdot T = I(C_2)$, where $I(C_i)$ is the identity functor on C_i .*

Proof: We have $UT(S, f) = (f(S) \oplus_\phi \text{Ker } f, f) = (S, f)$ by lemmas 3 and 4. Also, $TU(R, M, \phi) = T(R \oplus_\phi M, f) = (R, M, \phi)$ by lemmas 2 and 4. Furthermore, if $\alpha: (S, f) \rightarrow (S', f')$ is a morphism of C_1 , then $UT(\alpha) = U(\alpha|f(S), \alpha|\text{Ker } f) = \alpha^*$, where $\alpha^*(f(x) + (x - f(x))) = \alpha(x)$, i. e., $UT(\alpha) = \alpha$.

Finally, if $(\beta, \beta^*): (R, M, \phi) \rightarrow (R', M', \phi')$ is a morphism of C_2 , then $TU(\beta, \beta^*) = T(\alpha)$, where $\alpha(r+m) = \beta(r) + \beta^*(m)$ and $T(\alpha) = (\alpha|R, \alpha|M) = (\beta, \beta^*)$, i. e., $TU(\beta, \beta^*) = (\beta, \beta^*)$ and the conclusion follows.

Thus according to theorem 3, the notions of a semi-direct sum and of a decomposition are equivalent in the functorial sense indicated.

If C is the category of rings and homomorphisms, then the functor $F: C \rightarrow C_1$ given by $F(S) = (S, 1_s)$, where $1_s: S \rightarrow S$ is the identity map, and $f(\alpha) = \alpha$ for a homomorphism $\alpha: S \rightarrow S'$ embeds C in C_1 as a full subcategory. Indeed, if $\alpha: S \rightarrow S'$ is any homomorphism, then $\alpha: (S, 1_s) \rightarrow (S', 1_{s'})$ is a morphism in C_1 , since $\alpha \cdot 1_{s'} = 1_{s'} \cdot \alpha$. The functor $TF: C \rightarrow C_2$ associates

with S the triple $(S, 0, \phi)$, where $\phi: \{0\}^2 \rightarrow S \oplus 0$ is the multiplication.

Another way to embed C in C_1 is to use the functor $G: C \rightarrow C_1$ given by $G(S) = (S, 0_s)$, where $0_s: S \rightarrow S$ is the zero map, and $G(\alpha) = \alpha$ for a homomorphism $\alpha: S \rightarrow S'$. Again, since $\alpha \cdot 0_s' = 0_s \cdot \alpha$ for any α , G is a functor and $G(C)$ is a full subcategory of C_1 . $TG: C \rightarrow C_2$ associates with S the triple $(0, S, \phi)$ where $\phi: S^2 \rightarrow 0 \oplus S$ is the multiplication.

The nature of the category C_1 or C_2 can of course be investigated in much greater detail, but for our purposes theorem 3 is quite sufficient.

4. Properties of decompositions

In this section we prove several propositions about decompositions needed in the rest of the paper. In addition we discuss some of the standard examples of decompositions.

The principle of idealization is an instance of a special type of decomposition. Here we have a ring R , an R -bimodule M and a multiplication $\phi: M^2 \rightarrow R \oplus M$ given by $\phi(m_1, m_2) = 0$. Thus $R \oplus_\phi M$ becomes a ring with the multiplication defined by $(r+m)(r'+m') = rr' + rm' + mr'$. Therefore in particular the associated decomposition $f: A \rightarrow A$, $A = R \oplus_\phi M$, is a ring homomorphism.

PROPOSITION 3: *Suppose that $f: A \rightarrow A$ is a decomposition such that $xy + f(x)f(y) = f(x)y + xf(y)$ for all $x, y \in A$.*

Then $A = R \oplus_\phi M$, where $\phi(m_1, m_2) = 0$ for all $m_1, m_2 \in M$. Conversely, if $A = R \oplus_\phi M$, where $\phi(m_1, m_2) = 0$ for all $m_1, m_2 \in M$, then the associated decomposition f satisfies the condition

$$xy + f(x)f(y) = f(x)y + xy(y) \text{ for all } x, y \in A.$$

Proof: Since $M = \text{Ker } f$, $m_1 m_2 + f(m_1)f(m_2) = m_1 m_2 = f(m_1)m_2 + m_1 f(m_2) = 0$, i. e., $A = R \oplus_\phi M$, where $\phi(m_1, m_2) = m_1 m_2 = 0$. For the converse, since $x - f(x) \in M$, $(x - f(x))(y - f(y)) = 0$ and the proposition follows.

Thus, by an *idealization* we shall mean a decomposition $f: A \rightarrow A$ such that $xy + f(x)f(y) = f(x)y + xf(y)$ for all $x, y \in A$. To prove that an idealization is a ring homomorphism we note that

$$f(xy) + f(x)f(y) = f(x)f(y) + f(x)f(y) \text{ and } f(xy) = f(x)f(y_\phi).$$

PROPOSITION 4: *If $A = R \oplus M$ with associated decomposition f , then M is an ideal if and only if $\phi(M^2) \subseteq M$ and this is so if and only if f is an endomorphism.*

Proof: If f is an endomorphism then $\text{Ker } f = M$ is an ideal and $\phi(M^2) \subseteq M$. The converse is equally obvious.

If R_1 and R_2 are rings, let $R=R_1\oplus R_2$ be the (ring) direct sum of R_1 and R_2 . Suppose M is an R_1 - R_2 module with $(r_2m)r_1=r_2(mr_1)$ for $r_i\in R_i$. Define an R -action on M by $(r_1, r_2)m=r_2m$ and $m(r_1, r_2)=mr_1$. Then M becomes an R -bimodule. If $A=R\oplus_\phi M$, with $\phi(M^2)=0$, then this is equivalent to taking A to consist of matrices of the form

$$a = \begin{pmatrix} r_1 & 0 \\ m & r_2 \end{pmatrix}$$

with $m\in M$ and $r_i\in R$ and the usual multiplication of matrices.

PROPOSITION 5: *If A is any ring and if $f : A \rightarrow A$ is an idealization, then f is a radical decomposition.*

Proof: Suppose $A=R\oplus_\phi M$ is the semi-direct sum corresponding to f . Since M is an ideal and since $m-m-m(-m)=m^2=0$ for all $m\in M$, $M\subseteq J$.

PROPOSITION 6: *If $A=R\oplus_\phi M$ is the semi-direct sum corresponding to an idealization f of A and if A is a ring with 1, then $1\in R$ and $JA=JR\oplus_\phi M$.*

Proof: JA is the set of all $s\in A$ such that $1-st$ is right invertible for all $t\in A$. Since $1=r+m$ implies $r=r^2+rm$, $r=r^2$ and $rm=0$. Hence $m=rm+m^2=m^2=0$, i. e., $r=1\in R$.

Suppose that $s\in A$ and $m\in M$. We need to show that $1-ms$ is right invertible. If $s=r+m'$, then since $mm'=0$, we have $1-m(r+m')=1-mr$. Also, $(mr)^2=0$, whence $(1-mr)(1+mr)=1$ and

$$(1-ms)(1+mf(s))=1-m(s-f(s))+(ms)(mf(s))=1.$$

If $r\in JR$, then $s=r'+m$ yields $1-rs=1-rr'-rm$, and since $1-rm'$ is right invertible in R with right inverse u say, $(1-rs)u=1-rmu$ whence $(1-rs)u(1+rmu)=1$ and $JR\subseteq JA$. Hence $JR\oplus_\phi M\subseteq JA$.

Now suppose $s=r+m\in JA$. Let $t=r'+m'$. Then $1-st=1-rr'-m''$, where $m''=rm'+mr'$ has a right inverse $u=a+b$, $a\in R$, $b\in M$. Therefore

$$(1-rm'-m'')(a+b)=(1-rr')a-m''a+(1-rr'')b=1,$$

i. e., $(1-rr')a=1$. Since r' is arbitrary, $r\in JR$, i. e., $s\in JR\oplus_\phi M$.

Given a ring A with 1, a decomposition f is *unitary* if $A=R\oplus_\phi M$, where R is a ring with identity and M is a unitary R -bimodule. Since $f(1f(y))=f(1)f(y)=f^2(y)=f(y)$ and since $f(f(y)1)=f(y)$, the element $f(1)$ is the identity of R .

PROPOSITION 7: *Suppose A is any ring with 1 and suppose $f : A \rightarrow A$ is any decomposition. Then f is a unitary decomposition if and only if $f(1)=1$.*

Proof: Let $A=R\oplus_\phi M$ be the semi-direct sum associated with f .

If $f(1)=1$, then M is obviously a unitary R -bimodule. On the other hand

if M is a unitary R -bimodule, then $f(1)m = mf(1) = m$, and thus $(1-f(1))(x-f(x)) = 0 = x-f(x) - f(1)x + f(x)$, i.e., $x = f(1)x$. Similarly, $(x-f(x))(1-f(1)) = 0$ implies $x = xf(1)$. Hence $f(1) = 1$.

In particular any idealization is a unitary decomposition.

PROPOSITION 8: Suppose A is a ring with 1. If $f: A \rightarrow A$ is a radical decomposition, then f is a unitary decomposition.

Proof: If $A = R \oplus_\phi M$, then $1 = r + m$, $m \in M \subseteq JA$, implies $1 - m = r$ is right invertible, say $(1 - m)u = ru = u - mu = 1 = r + m$. Let $u = r' + m'$, then $r' - r = mu + m$, whence $r' = r$, $u = r + m'$. Now, $(1 - m)(r + m') = r(r + m') = 1 = r^2 + rm'$, i.e., $r^2 = r$ and $m = rm'$.

Since $1 = r(1 + m')$, $(1 - r)r(1 + m') = 0 = 1 - r$, i.e., $r = 1$ and $m = 0$. Thus $f(1) = 1$ and f is a unitary decomposition as asserted.

PROPOSITION 9: Suppose A is a ring with 1, and suppose f is a radical decomposition. If $A = R \oplus_\phi M$ is the associated semi-direct sum, then

$$JA = JR \oplus_\phi M.$$

Proof: The proof is quite similar to the proof of proposition 6, Suppose that $r \in JR$. Let $s = r' + m \in A$. Then, $1 - rs = 1 - r(r' + m) = 1 - rr' - rm$, and since $r \in JR$, $1 - rr'$ is right invertible in R , i.e., $(1 - rr')u = 1$ for some $u \in R$. Therefore $(1 - rs)u = (1 - rr')u - rm u = 1 - rm u$, and since $rm u \in M \subseteq JA$, $1 - rm u$ is invertible in A , i.e., $(1 - rs)us' = 1$ for some $s' \in A$. Hence $r \in JA$ and $JR \oplus_\phi M \subseteq JA$.

Suppose now that $s = r + m \in JA$. Let $t = r' + m'$. Then, $1 - st = 1 - rr' - m''$, where $m'' = rm' + mr' + mm'$. Now $1 - st$ has a right inverse $u = a + b$, $a \in R$, $b \in M$. Hence, $(1 - rr' - m'')(a + b) = (1 - rr')a - m''a + (1 - rr')b = 1$ and since $1 \in R$, $(1 - rr')a = 1$. Since r' is arbitrary, $r \in JR$, and $s \in JR \oplus_\phi M$.

PROPOSITION 10: Suppose that $A = R \oplus_\phi M$ is the semi-direct sum associated with an idealization of A . Suppose that R is a ring with 1, JR is T -nilpotent and M is a unitary R -bimodule. Then A is a ring with 1 and JA is T -nilpotent.

Proof: Since $1(r + m) = r + 1m = r + m$ and $(r + m)1 = r + m$, A has an identity 1 and $f(1) = 1$, where f is the associated decomposition. By proposition 6, f is a radical decomposition and $JA \oplus_\phi M$.

Consider the sequence $\{r_i + m_i\} \subseteq JA$, where $r_i \in JR$, $m_i \in M$. Then $(r_k + m_k) \dots (r_1 + m_1) = r_k \dots r_1 + m_k r_{k-1} \dots r_1 + \dots + r_k \dots r_2 m_1$, since all other terms end up in $M^2 = 0$.

Pick t such that $r_{t-1}r_{t-2} \dots r_1 = 0$ and t' such that $r_{t'}, \dots r_{t'+1} = 0$. Then, for $k = t'$, $(r_k + m_k) \dots (r_1 + m_1) = 0$, whence JA is T -nilpotent.

PROPOSITION 11: *Suppose that A is a ring with 1, and suppose that $A = R \oplus_{\phi} M$ is the semi-direct sum associated with a radical decomposition. Then $A/JA = R/JR$.*

Proof: Since $A = R \oplus_{\phi} M$ and $JA = JR \oplus_{\phi} M$, it follows that the mapping $(r+m) + JA \rightarrow r + JR$ is an additive isomorphism.

Also $(r+m)(r'+m') + JA \rightarrow rr' + a + JR'$ where $mm' = a + b$, $a \in R$, $b \in M$. Since $mm' \in JA = JR \oplus_{\phi} M$, $a \in JR$ and $(r+m)(r'+m') + JA \rightarrow rr' + JR$, i. e., the mapping is also multiplicative.

COROLLARY 1: *If $A = R \oplus_{\phi} M$ is the semi-direct sum associated with a radical decomposition and if R/JR is Artinian then A/JA is Artinian.*

COROLLARY b: *If $A = R \oplus_{\phi} M$ is the semi-direct sum associated with a radical decomposition and if R is semi-perfect, then A is semi-perfect.* *

COROLLARY 3: *If $A = R \oplus_{\phi} M$ is the semi-direct sum associated with a radical decomposition and if R is perfect, then A is perfect.*

COROLLARY 4: *If $A = R \oplus_{\phi} M$ is the semi-direct sum associated with an idealization and if R is perfect then A is perfect.*

Say R is *indecomposable* if the only decompositions of R are 0_R and 1_R . Thus \mathbf{Z} , the ring of integers is indecomposable. If \mathbf{Q} is the field of rationals the same is true. If p is any prime number then the Steinitz ring $\mathbf{Z}/(p^i)$, $i \geq 1$, has the same property. If F is any field, and if K is its prime subfield, $F = K \oplus_{\phi} M$, where M is a K -vector space. Thus a field is indecomposable if and only if it is a prime field. There are simple rings which are decomposable and indecomposable rings which are not semisimple. No group ring $R[G]$ with $G \neq \langle 1 \rangle$ is indecomposable. Indeed, $tr^* : R[G] \rightarrow R[G]$ defined by $tr^*(x) = tr(x)1$, is a decomposition, where $tr(x)$ is the trace map. Similarly, $f : R[X] \rightarrow R[X]$ defined by $f(P(X)) = P(0)$ is a decomposition.

Since the concept of indecomposability will not be needed in the rest of this paper we have not attempted an in depth discussion, However, the observations made above do indicate that indecomposable rings are sufficiently scarce that an attempt to catalogue them might prove interesting.

5. Normal rings and L. C. I. rings

If A is any ring, then CA is the center of A . A ring R is *normal* if $C(R/JR) = (CR + JR)/JR$. Thus if R is normal, then the elements in the center of R/JR can be lifted to the center of R . Semisimple and commutative rings are normal. If R is normal, then if JR is a nil ideal, central idempotents of R/JR can be lifted to central idempotents of R . A ring R

is a ring suitable for lifting central idempotents modulo the Jacobson radical (an *L.C.I. ring*) if central idempotents of R/JR can be lifted to central idempotents of R .

By corollary 1 of theorem 1 it follows that if R is perfect and an *L.C.I. ring*, then R is central perfect. If R is central perfect on the other hand, then by the same corollary it follows that R is an *L.C.I. ring*. From the observations made above it follows that if R is normal and perfect it is a perfect *L.C.I. ring* and hence a central perfect ring.

In the sections concerned with group rings we shall consider the problem of determining normal perfect group rings and relate this to the problem of determining which group rings are central perfect. For this reason as well as to give many examples we shall look at normal rings and *L.C.I. rings* in somewhat more detail in this section.

EXAMPLE 2: Not every *L.C.I. ring* is normal. In fact not every Steinitz ring is normal, Furthermore subrings of normal rings need not be normal. We give some constructions which demonstrate these statements and which enlarge the class of examples somewhat.

If R is a perfect ring and if R_n is the ring of $n \times n$ matrices with coefficients in R , then R_n is perfect. Also, if $T_n(R)$ is the ring of lower triangular matrices in R_n then $T_n(R)$ is perfect. If $\tau_n(R)$ denotes the subring of $T_n(R)$ consisting of those matrices with constant diagonal, then $J\tau_n(R) \subseteq JT_n(R)$, i.e., $J\tau_n(R)$ is T -nilpotent. Also, $\tau_n(R)/J\tau_n(R) = R/JR$ which is semisimple Artinian. Thus $\tau_n(R)$ is perfect. Similarly, if $\tau_\infty(R)$ consists of column-finite row-finite matrices of the lower triangular type with constant diagonal, then $J\tau_\infty(R)$ is T -nilpotent since it consists of matrices $\lambda J_\infty + X$, where $X \in \tau_\infty(R)$, $X_{ii} = 0$ for all i and $\lambda \in JR$. Again, $\tau_\infty(R)/JT_\infty(R) = R/JR$. In fact, the mapping of elements of $\tau_n(R)$ or $\tau_\infty(R)$ to their diagonals is a radical decomposition f with image a ring isomorphic to R .

Thus, if R is a Steinitz ring, then $\tau_n(R)$ and $\tau_\infty(R)$ are also Steinitz rings.

Now let $\sigma = (\sigma_1, \dots, \sigma_n)$ be a sequence of automorphisms of R . Then by $\tau_n(R; \sigma)$ we shall denote the subring of $T_n(R)$ consisting of matrices X such that for some $a \in R$, $X_{ii} = a^{\sigma_i}$. Thus, if $\sigma = \sigma_2 = \dots = \sigma_n = 1_R$, then $\tau_n(R; \sigma) = \tau_n(R)$. It follows that $\tau_n(R; \sigma)$ is indeed a ring. If N consists of all matrices $X \in \tau_n(R; \sigma)$ with $X_{ii} = 0$, then N is an ideal and $N^n = 0$, whence $N \subseteq J\tau_n(R; \sigma)$. Suppose that $f: \tau_n(R; \sigma) \rightarrow \tau_n(R; \sigma)$ is given by $f(X) = Y$, where $X_{ii} = Y_{ii}$ and $Y_{ij} = 0$ if $i \neq j$. Then f is a radical decomposition since $N = \text{Ker } f$. Also, $f(\tau_n(R; \sigma)) = R^*$ is isomorphic to R and if $R^* \oplus_\phi N$ corresponds to the decomposition f , then $J\tau_n(R; \sigma) = JR^* \oplus_\phi N$ by proposition 9

and $\tau_n(R; \sigma)/J\tau_n(R, \sigma) = R^*/JR^* = R/JR$ by proposition 11. If JR is T -nilpotent, then by proposition 10, $J\tau_n(R; \sigma)$ is T -nilpotent. Hence, if R is perfect, $\tau_n(R; \sigma)$ is perfect, while if R is Steinitz, $\tau_n(R; \sigma)$ is also Steinitz. Clearly, the same construction works for $\tau_\infty(R; \sigma)$.

Furthermore, if we take $\sigma = (\sigma_1, \dots, \sigma_n)$ to be a sequence of endomorphisms with $\cap \text{Ker } \sigma_i = 0$ and $\sigma_i(1) = 1$, then $\tau_n(R; \sigma)$ is a subring of $T_n(R)$ and $f(\tau_n(R; \sigma)) = R^*$ is isomorphic to R , since the mapping $X \rightarrow a$, where $X_{ii} = a^{\sigma_i}$ and $X_{ij} = 0$ if $i \neq j$, is an injection and since the mapping is obviously an epimorphism. Again, if R is perfect then $\tau_n(R; \sigma)$ is perfect and if R is Steinitz then $\tau_n(R; \sigma)$ is Steinitz. Suppose now that F is a field with a nontrivial automorphism τ . Let K be the fixed field of τ . Then $K \neq F$. If $\sigma = (1, \tau)$, where 1 denotes the identity map, then $\tau_2(F; \sigma)$ is a Steinitz ring, $\tau_2(F; \sigma) = F^* \oplus_\phi N$, $J\tau_2(F; \sigma) = JF^* \oplus_\phi N = N$ and $\tau_2(F; \sigma)/J\tau_2(F; \sigma) = F^*/F$ is a commutative ring.

If $X \in C\tau_2(F; \sigma)$ and if $Y_{21} = Y_{12} = 0$, $Y_{11} = a$, $Y_{22} = a^\tau$, with $a \in F/K$, then $XY = YX$ implies $X_{21} = X_{12} = 0$. If E_{21} is the matrix with $(E_{21})_{ij} = \delta_{2i}\delta_{1j}$, then $XE_{21} = E_{21}X$ implies $a = a^\tau$, whence $a \in K \neq F$. Thus $C\tau_2(F; \sigma) = K^*$ and $C\tau_2(F; \sigma) + J\tau_2(F; \sigma)/J\tau_2(F; \sigma) = K^*/K \neq F$ and $\tau_2(F; \sigma)$ is not a normal ring. Since F_2 is semisimple it is a normal ring. Hence $\tau_2(F; \sigma)$ is a subring of a normal ring which is not itself normal.

The example of a Steinitz ring which is not normal is due to D. S. Passman.

PROPOSITION 12: *Suppose R is an L. C. I. ring. Let $\sigma = (\sigma_1, \sigma_2, \dots)$ where σ_i maps central idempotents to central idempotents and such that if e is a central idempotent of R , then $e^{\sigma_1} = e^{\sigma_i}$ for all i .*

If $A = \tau_n(R; \sigma)$ or $A = \tau_\infty(R; \sigma)$, then A is an L. C. I. ring.

Proof: Let f be the decomposition of A which maps matrices to their diagonals as in example 2. Then f is a radical decomposition and $f(A) = R^*$, $A = R^* \oplus_\phi N$. Furthermore, the mapping $\theta: R \rightarrow R^*$ given by $\theta(a) = X$, where X is the diagonal matrix with $X_{ii} = a^{\sigma_i}$, is an isomorphism.

Let \bar{e} be a central idempotent of R/JR which is lifted to the central idempotent e of R and mapped to the element $X = \theta(e)$. It follows that $X = e^{\sigma_1}I$, which is clearly a central idempotent with $X + JA = \bar{e}$ in $A/JA = R/JR$ as in proposition 11.

COROLLARY 1: *If R is an L. C. I. ring and if $T_n(R)$ is the ring of lower triangular matrices with coefficients in R , then $T_n(R)$ is an L. C. I. ring.*

Proof: Let \bar{e} be a central idempotent of R/JR which is lifted to the central idempotent e of R and then mapped to the diagonal matrix eI of $T_n(R)$. The latter is a central idempotent as in proposition 12, with $eI + JT_n(R) = \bar{e}$.

COROLLARY 2: *If R is an L.C.I. ring and if R_n is the complete ring of $n \times n$ matrices with coefficients in R , then R_n is an L.C.I. ring.*

Proof: Let \bar{e} be a central idempotent of R/JR which is lifted to the central idempotent e of R and then mapped to the diagonal matrix eI or R_n . As in proposition 12, the latter is a central idempotent with $eI + JR_n = \bar{e}I$, $\bar{I} = I + JR_n$.

Since $R_n/JR_n = (R/JR)_n$, any central idempotent has the form $\bar{e}I$ for some central idempotent \bar{e} of R/JR . The conclusion follows.

PROPOSITION 13: *If A and B are L.C.I. rings, then $R = A \oplus B$ is an L.C.I. ring.*

Proof: The proposition follows since

$$JR = JA \oplus JB \text{ and } R/JR = A/JA \oplus B/JB.$$

COROLLARY 1: *Let A_1, A_2, \dots, A_n be a family of L.C.I. rings. Then $R = (A_1)_{m_1} \oplus \dots \oplus (A_n)_{m_n}$ is an L.C.I. ring.*

Since every local ring is an L.C.I. ring, it follows that rings which are complete matrix rings over local rings and finite direct sums of rings of this type are L.C.I. rings. Replacing the fact that JR is nil by the assumption that idempotents can be lifted modulo the radical and using the same constructions as in theorems 1 and 2 and corollaries, with the rings $e_i R e_i$ local rings instead of Steinitz rings, it follows that a semiperfect ring is an L.C.I. ring if and only if it is a finite direct sum of complete matrix rings over local rings. Furthermore, using the notation e_1, \dots, e_t for local idempotents and f_1, \dots, f_n for central idempotents, we have for semiperfect rings a radical decomposition $f: R \rightarrow R$ given by $f(r) = r^* = \sum_{i=1}^n f_i r f_i$, with $\text{Ker } f = 0$ if and only if R is an L.C.I. ring. Finally, if $g: R \rightarrow R$ is any radical decomposition, then there is an automorphism α of R such that $(\alpha \cdot g \cdot \alpha^{-1})(R) \subseteq f(R)$. Furthermore, any two decompositions of the type $f: R \rightarrow R$, corresponding to standard normal decompositions, are conjugate by Azumaya's theorem, as in the proof of theorem 2.

Suppose that R is a ring with 1 and $f: R \rightarrow R$ is a radical decomposition such that $f(R)$ is a normal ring, then R is *almost normal*.

PROPOSITION 14: *If R is a normal ring, then $\tau_n(R; \sigma)$ and $\tau_\infty(R; \sigma)$ are almost normal rings.*

Proof: Let f be the decomposition which maps matrices to diagonal matrices by setting the off-diagonal elements equal to 0 as above. Then f is a radical decomposition with image isomorphic to R , a normal ring.

If $N = \text{Ker } f$, then in $\tau_n(R; \sigma)$, N is nilpotent. For $\tau_\infty(R; \sigma)$, define A_n as the matrix obtained by setting $A_{ij} = 0$ if $i, j > n$ in the matrix A . Solve for $B_n = B_1 + (B_2 - B_1) + (B_3 - B_2) + \dots$

Since $A = A_1 + (A_2 - A_1) + (A_3 - A_2) + \dots$, it follows that $A - B + AB = 0$, so that N is a quasi-regular ideal, i.e., $N \subseteq J(\tau_\infty(R; \sigma))$.

Thus in example 2 all Steinitz rings constructed are almost normal if one starts with normal rings.

PROPOSITION 15: *If R is a ring with 1 and if $f: R \rightarrow R$ is a radical decomposition such that $f(R)$ is almost normal, then R is almost normal.*

Proof: Suppose $g: A = f(R) \rightarrow A$ is a radical decomposition with $T = g(A)$ a normal ring. Define $h: R \rightarrow R$ by letting $h(x) = gf(x)$. It follows that h is a decomposition. By using proposition 9 twice, we find that

$$JR = JA \oplus_{\phi} \text{Ker } f \text{ and } JA = JT \oplus_{\phi} \text{Ker } g.$$

Hence we have a group direct sum $JR = JT \oplus \text{Ker } g \oplus \text{Ker } f$. Again, $\text{Ker } h = \text{Ker } g \oplus \text{Ker } f \subseteq JR$ whence h is a radical decomposition, and $h(R) = T$ is a normal ring, so that R is almost normal.

COROLLARY 1: *Suppose that R is an almost normal ring. Then any ring $\tau_n(R; \sigma)$ or $\tau_\infty(R; \sigma)$ is an almost normal ring.*

PROPOSITION 16: *If $\tau_n(R; \sigma)$ or $\tau_\infty(R; \sigma)$ is a normal ring, then R is a normal ring.*

Proof: If $n=1$, there is no problem since $\tau_1(R; \sigma) = R^* = R$. For $n > 1$, $C\tau_n(R; \sigma)$ is the collection of all matrices aI_n , where $a^{\sigma_1} = a^{\sigma_2} = \dots = a^{\sigma_n}$, and $a \in CR$. Thus, if C^*R denotes this subring, $(C^*R)I_n + J(\tau_n(R; \sigma)) / J(\tau_n(R; \sigma)) = C(R/JR)$, whence $C^*R + JR/JR = C(R/JR)$.

Since $C^*R \subseteq CR$, $CR + JR/JR = C(R/JR)$ and R is a normal ring. The same computation holds for $\tau_\infty(R; \sigma)$.

EXAMPLE 3: If R is a normal ring, then $T_n(R)$ is not normal in general. Indeed, let R be a commutative ring and let X be a matrix in $T_n(R)$ which has a diagonal whose elements are not constant modulo JR . It follows that in this situation there is no matrix Y in $CT_n(R) + JT_n(R)$ which is congruent to X modulo $JT_n(R)$. Thus $T_n(R)$ is not normal.

PROPOSITION 17: *If $R = A \oplus B$ is the ring direct sum of A and B , then R is normal if and only if both A and B are normal. Also, R is almost normal if both A and B are almost normal.*

Proof: Since $CR = CA \oplus CB$, $JR = JA \oplus JB$, $CR + JR/JR = (CA + JA/JA) \oplus (CB + JB/JB) = C(A/JA) \oplus C(B/JB) = C(A/JA \oplus B/JB) = C(R/JR)$, if A

and B are both normal.

Conversely, if R is normal, let $a+JA=a+JR$ be an element of $C(R/JR)$. If $x+JR$ maps to $a+JA$, with $x \in CR$, then $x=\alpha+\beta$, $\alpha \in CA$, $\beta \in CB$. Thus we have $x+JR=(\alpha+JA)+(\beta+JB)$, and $x+JR=a+JR$ implies $\beta+JB=0$, i.e., $\beta \in JB$, whence we may take $x=\alpha \in CA$, whence $CA+JA/JA=C(A/JA)$, i.e., A is normal. Similarly B is normal. If A and B are almost normal, let $g_A: A \rightarrow A$ and $g_B: B \rightarrow B$ be radical decompositions so that $g_A(A)$ and $g_B(B)$ are normal. Define $f: R \rightarrow R$ by $f(a+b)=g_A(a)+g_B(b)$, where $a \in A$, $b \in B$. Then f is a decomposition with $f(R)=g_A(A)+g_B(B)$ a normal ring. Since $\text{Ker } f = \text{Ker } g_A \oplus \text{Ker } g_B \subseteq JA \oplus JB = JR$, f is a radical decomposition and R is an almost normal ring.

PROPOSITION 18: Suppose that R is an almost normal ring, then $T_n(R)$ is almost normal.

Proof: Let $f: T_n(R) \rightarrow T_n(R)$ be the radical decomposition which maps matrices to their diagonals. Then $f(T_n(R))=R \oplus \cdots \oplus R$ is almost normal and by proposition 15, $T_n(R)$ is almost normal.

PROPOSITION 19: The ring R_n is normal if and only if R is normal. Furthermore, if R is almost normal, then R_n is almost normal as well.

Proof: Note that $CR_n=\{aI_n \mid a \in CR\}$, $JR_n=(JR)_n$. Thus $C(R_n/JR_n)=C((R/JR)_n)$ consists of matrices $(a+JR)(I_n+JR_n)$ with $a+JR \in C(R/JR)$. If R is normal, $a=\alpha+\beta$, $\alpha \in CR$, $\beta \in JR$ and $\alpha I_n+\beta I_n \in CR_n+JR_n$, whence $CR_n+JR_n/JR_n=C(R_n/JR_n)$ and R_n is normal. If R_n is a normal ring, then $a+JR \in C(R/JR)$ implies $(a+JR)(I_n+JR_n) \in C(R_n/JR_n)$, whence $aI_n=\alpha I_n+B$, $\alpha I_n \in CR_n$, $B \in JR_n$, i.e., $B=(a-\alpha)I_n=\beta I_n$ and $\beta=a-\alpha \in JR$. Hence $CR+JR/JR=C(R/JR)$ and R is normal.

Suppose R is almost normal. Let $f: R \rightarrow R$ be a radical decomposition with $f(R)$ normal. Define $f_n: R_n \rightarrow R_n$ by $f_n(X)=Y$ with $Y_{ij}=f(X_{ij})$. Then $X-f_n(X) \in (JR)_n=JR_n$ and $f_n(R_n)=(f(R))_n$, which is a normal ring.

The mapping f_n is additive and idempotent and $\text{Ker } f \subseteq JR_n$, i.e., f_n is a radical decomposition provided we can prove the multiplicative property.

Suppose $X \in f_n(R_n)$, i.e., $X_{ij} \in f(R)$. Then if $XY=Z$, $Z_{ij}=\sum_k X_{ik}Y_{kj}$ and $f(Z_{ij})=\sum_k f(X_{ik})f(Y_{kj})=\sum_k f_n(X)_{ik}f_n(Y)_{kj}$. So that $f(XY)=f_n(Z)=f_n(X)f_n(Y)$. Similarly, if $Y \in f_n(R_n)$, then $f_n(XY)=f_n(X)f_n(Y)$, whence f_n is indeed a radical decomposition and R_n is almost normal.

COROLLARY 1: If S is a normal or almost normal Steinitz ring, then S_n is a normal or almost normal perfect ring. Conversely if S_n is the complete ring

of $n \times n$ matrices over a Steinitz ring S and if S_n is normal perfect, then S is normal perfect.

PROPOSITION 20: *If R is a normal ring, JR a nil ideal and if R/JR has a complete orthogonal set of central idempotents $\{\bar{e}_1, \dots, \bar{e}_k\}$, then this set may be lifted to a complete orthogonal set of central idempotents $\{e_1, \dots, e_k\}$ of R .*

Proof: Suppose $1 = \bar{e}_1 + \dots + \bar{e}_k$, where $e_i \in C(R/JR)$. Since R is normal there is an element α_i of R such that $\alpha_i + JR = \bar{e}_i$, i. e., $\alpha_i^2 - \alpha_i \in JR$.

Using the standard trick, if we let $e_i = \alpha_i + x(1 - 2\alpha_i)$, where $x = \frac{1}{2}(1 - (1 + 4n)^{-1/2})$, $n = \alpha_i^2 - \alpha_i$, then e_i is an idempotent, $e_i + JR = \bar{e}_i$, where since $\alpha_i \in CR$, $e_i \in CR$ as well. Since $e_i e_j \in JR$, $(e_i e_j)^t = e_i e_j = 0$ for some integer t , and $e_1 + \dots + e_k = u$ is a central idempotent with $1 - u \in JR$, whence $u = 1$, i. e., $\{e_1, \dots, e_k\}$ is a complete orthogonal set of central idempotents as asserted.

PROPOSITION 21: *If R is a normal perfect ring it is central perfect*

Proof: If $R/JR = (D_1)_{m_1} \oplus \dots \oplus (D_n)_{m_n}$, with $(D_i)_{m_i} = \bar{f}_i(R/JR) = (R/JR)\bar{f}_i$, $\bar{f}_i \bar{f}_j = \delta_{ij}$, then $\{f_1, \dots, f_n\}$ is a complete orthogonal set of centrally primitive idempotents and since JR is a nil ideal we may lift this to a complete orthogonal set $\{f_1, \dots, f_n\}$ of central idempotents. Thus the standard normal decomposition $f: R \rightarrow R$ given by $f(r) = \sum f_i r f_i$ is the identity map, whence R is central perfect as asserted.

COROLLARY 1: *If R is a normal perfect ring, then all Steinitz rings which appear in the representation of R are normal and conversely.*

Proof: Since R is central perfect by proposition 21, $R = (S_1)_{m_1} \oplus \dots \oplus (S_n)_{m_n}$. By propositions 17 and 19 it follows that the S_i are normal Steinitz rings. The converse follows in the same way.

By propositions 17 and 19 it follows also that if $R^* = (S_1)_{m_1} \oplus \dots \oplus (S_n)_{m_n}$, where all the Steinitz rings are almost normal, then R^* is almost normal. Thus if $f: R \rightarrow R$ is a normal decomposition of the perfect ring R with $f(R) = R^* = (S_1)_{m_1} \oplus \dots \oplus (S_n)_{m_n}$ and all the Steinitz rings almost normal, then R is almost normal.

Group Rings

We shall be concerned with the following problems: classify R and G if $R[G]$ is: (1) L. C. I.; (2) normal; (3) almost normal; (4) normal perfect; (5) almost normal perfect; (6) central perfect; (7) normal Steinitz; (8) almost normal Steinitz.

To construct some nontrivial group-ring examples we prove the following.

PROPOSITION 22: Suppose R is a ring of characteristic $p^i > 0$ and suppose that G is a finite p -group. Then if R is L.C.I. (resp. normal, almost normal, normal perfect, almost normal perfect, central perfect, normal Steinitz, almost normal Steinitz), it follows that $R[G]$ is L.C.I. (resp. normal, almost normal, normal perfect, almost normal perfect, central perfect, normal Steinitz, almost normal Steinitz).

Proof: Let us consider the normhomomorphism $N(\alpha) = N \sum \alpha(g)g = \sum \alpha(g)$. Then N is a decomposition with $\text{Ker } N$, the fundamental ideal, contained in $JR[G]$, i.e., N is a radical decomposition. Thus, $R[G]/JR[G] = R/JR$.

Suppose that R is an L.C.I. ring. If \bar{e} is a central idempotent of R/JR , lift \bar{e} to a central idempotent e of R . Then e is a central idempotent of $R[G]$ and $e + JR[G] = e + JR$ in the isomorphism of the first paragraph. Thus $R[G]$ is an L.C.I. ring.

If R is normal, suppose that $\alpha JR[G] \in C(R/JR)$. Then since $\alpha + JR[G] = N(\alpha) + JR[G] = N(\alpha) + JR$ (in R/JR), there are elements $\beta \in CR$, $\gamma \in JR$, with $N(\alpha) = \beta + \gamma$. Hence $\alpha + JR[G] = \beta + JR[G]$ and

$$CR[G] + JR[G]/JR[G] = C(R/JR) = C(r[G]/JR[G]).$$

Thus $R[G]$ is a normal ring.

Suppose that R is almost normal. Then since $N: R[G] \rightarrow R[G]$ is a radical decomposition with $N(R[G]) = \text{almost normal}$, it follows by proposition 15 that $R[G]$ is almost normal.

Since $R[G]$ is perfect if and only if R is perfect and G is finite, the proposition follows for R normal perfect or almost normal perfect.

If R is Steinitz, then $R[G]$ is Steinitz and conversely. Hence the proposition follows for normal Steinitz rings and almost normal Steinitz rings. If R is central perfect, then

$$R = (S_1)_{m_1} + \cdots + (S_n)_{m_n} \text{ and } R[G] = (S_1[G])_{m_1} + \cdots + (S_n[G])_{m_n},$$

as we show in the next proposition. If R has characteristic $p^i > 0$, then S_j has characteristic $p^{i(j)} > 0$, whence $S_j[G]$ is a Steinitz ring. Hence $R[G]$ is central perfect.

PROPOSITION 23: Given a ring R and a group G we have the following isomorphisms: $R_n[G] = (R[G])_n$, $T_n(R)[G] = T_n(R[G])$,

$$\tau_n(R; \sigma)[G] = \tau_n(R[G]; \sigma^*), \quad \tau_\infty(R; \sigma)[G] = \tau_\infty(R[G]; \sigma^*),$$

where $\sigma^* = (\sigma_1^*, \sigma_2^*, \dots)$ and $\sigma_i^*(\alpha) = \sum \alpha(g) \sigma_i g = \alpha^{\sigma_i^*}$.

Proof: Note that if σ_i is an endomorphism of R , then σ_i^* is an endomorphism of $R[G]$ with $\text{Ker } \sigma_i^* = (\text{Ker } \sigma_i)[G]$, whence $\cap \text{Ker } \sigma_i = 0$ implies $\cap \text{Ker } \sigma_i^* = 0$.

Suppose $\alpha \in R_n[G]$. Then $\alpha = \sum \alpha(g)g$, $\alpha(g) \in R_n$. Define $\alpha_{ij} \in R[G]$ by $\alpha_{ij} = \sum \alpha(g)_{ij}g$. Now, we have $\alpha\beta = \sum \alpha(g)\beta(h)gh$, where $(\alpha(g)\beta(h))_{ij} =$

$\sum_k \alpha(g)_{ik} \beta(h)_{kj}$, whence $\alpha \beta_{ij} = \sum_{g,h} (\alpha(g) \beta(h))_{ij} gh = \sum_{g,h} (\sum_k \alpha(g)_{ik} \beta(h)_{kj}) gh = \sum_k (\sum_{g,h} \alpha(g)_{ik} \beta(h)_{kj} gh) = \sum_k (\sum_g \alpha(g)_{ik} g) (\sum_h \beta(h)_{kj} h) = \sum_k \alpha_{ik} \beta_{kj}$. Hence the mapping $\alpha \rightarrow (\alpha_{ij})$ has the property that $\alpha \beta \rightarrow (\alpha \beta)_{ij} = (\alpha_{ik}) (\beta_{kj})$, so that since the mapping is obviously both an epimorphism and a monomorphism we have $R_n[G] = (R[G])_n$.

If $\alpha \in T_n(R)[G]$, then $\alpha = \sum \alpha(g)g$, with $\alpha(g)_{ij} = 0$ if $j > i$. Hence if we map $\alpha \rightarrow (\alpha_{ij})$ as above, then if $j > i$, $\alpha_{ij} = \sum \alpha(g)_{ij} g = 0$, whence $(\alpha_{ij}) \in T_n(R[G])$. The isomorphism $T_n(R)[G] = T_n(R[G])$ is now clearly given by the restriction of the isomorphism $R_n[G] = (R[G])_n$ to $T_n(R)[G]$.

If $\alpha \in \tau_n(R; \sigma)[G]$, then $\alpha = \sum \alpha(g)g$, with $\alpha(g)_{ij} = 0$ if $j > i$ and $\alpha(g)_{ii} = \lambda(g)^{\sigma_i}$ for some $\lambda \in R$. Thus $\alpha_{ii} = \sum \alpha(g)_{ii} g = \sum \lambda(g)^{\sigma_i} g = (\sum \lambda(g)g)^{\sigma_i^*}$, with $\sum \lambda(g)g \in R[G]$.

Thus the isomorphism $\tau_n(R; \sigma)[G] = \tau_n(R[G]; \sigma^*)$ is given by the restriction of the mapping $\alpha \rightarrow (\alpha_{ij})$ given above.

The construction $\tau_\infty(R; \sigma)[G] = \tau_\infty(R[G]; \sigma^*)$ is the same as the construction for $\tau_n(R; \sigma)[G] = \tau(R[G]; \sigma^*)$.

PROPOSITION 24: *If R is a semiperfect L.C.I. ring, and if T is an epimorphic image of R , then T is a semiperfect L.C.I. ring.*

Proof: If R is a semiperfect L.C.I. ring, then $R = (S_{1m_1} \oplus \cdots \oplus (S_n)_{m_n})$, where the S_i are local rings, and hence themselves semiperfect L.C.I. rings. If I is any ideal of R , then $I = (I_1)_{m_1} \oplus \cdots \oplus (I_n)_{m_n}$, where I_i is an ideal of S_i , and where S_i/I_i is itself a local ring.

Thus $T = R/I = (S_1/I_1)_{m_1} \oplus \cdots \oplus (S_n/I_n)_{m_n}$, and the proposition follows from proposition 13 and its corollary 1 as well as the discussion following that corollary.

COROLLARY 1: *If $R[G]$ is a semiperfect L.C.I. ring so is R .*

COROLLARY 2: *If $R[G]$ is a central perfect ring (i.e., a perfect L.C.I. ring), then R is a central perfect ring.*

Proof: The epimorphic image of a Steinitz ring (local perfect) ring is a Steinitz ring. The result follows from proposition 24.

PROPOSITION 25: *If R is a normal local ring and if T is an epimorphic image of R , then T is a normal local ring.*

Proof: If $T = R/I$, then since $I \subseteq JR$, we have $R/JR = (R/I)/(JR/I) = (R/I)/J(R/I)$, since R/JR is a division ring. Now, if $\bar{x} \in C(R/JR)$ and $x \in CR$, with $x + JR = \bar{x}$, then $x + I = x^*$ yields $x^* + J(R/I) = \bar{x}$ and $x^* \in C(R/I)$, whence the conclusion follows.

COROLLARY 1: If $R[G]$ is a normal local ring, so is R .

COROLLARY 2: If $R[G]$ is a normal Steinitz ring so is R .

Proof: If $R[G]$ is perfect so is R . Hence corollary 2 follows from corollary 1.

PROPOSITION 26: If R is a normal semiperfect ring with JR a nil ideal and if T is an epimorphic image of R , the T is a normal semiperfect ring with JT a nil ideal.

Proof: Since R is a normal semiperfect ring, $R/JR = (D_1)_{m_1} \oplus \cdots \oplus (D_n)_{m_n}$, with a complete orthogonal set of centrally primitive idempotents $\bar{f}_1, \dots, \bar{f}_n$, which may be lifted to a complete orthogonal set of centrally primitive idempotents f_1, \dots, f_k of R , by proposition 20.

But then it follows that, using the argument following proposition 13, $R = (L_1)_{m_1} \oplus \cdots \oplus (L_n)_{m_n}$, where L_i is a local ring. Now, identifying L_i with the appropriate diagonal matrices, we have as in proposition 19, $C(L_i/JL_i) = CL_i + JL_i/JL_i$, whence the L_i are normal rings.

If $T = R/I$, $T = (L_1/I_1)_{m_1} \oplus \cdots \oplus (L_n/I_n)_{m_n}$, by proposition 25. If $I_1 \neq L_1$, ..., $I_k \neq L_k$, $I_{k+1} = L_{k+1}$, ..., $I_n = L_n$, then $T = (L_1/I_1)_{m_1} \oplus \cdots \oplus (L_k/I_k)_{m_k}$, a direct sum of complete matrix rings over normal local rings. By propositions 17 and 19 this is a normal semiperfect ring.

Since $JT = (JL_1/I_1)_{m_1} \oplus \cdots \oplus (JL_k/I_k)_{m_k}$, with $JR' = (JL_1)_{m_1} \oplus \cdots \oplus (JL_k)_{m_k}$ nil if JR is nil, it follows that JT is also nil, since $JT = JR'/I'$, where $I' \subseteq JR'$. The proposition follows.

COROLLARY 1: If $R[G]$ is a normal semiperfect ring with $JR[G]$ a nil ideal, then the same is true for R .

COROLLARY 2: If $R[G]$ is a normal perfect ring then R is also a normal perfect ring.

PROPOSITION 27: If R_n is a central perfectring, then R is central perfect and conversely.

Proof: If $R_n = (S_1)_{m_1} \oplus \cdots \oplus (S_k)_{m_k}$, then JS_i is T -nilpotent for each Steinitz ring S_i . Now there are ideals I_i of R such that $(R/I_i)_n = R_n/(I_i)_n = (S_i)_{m_i} = A_i$.

Hence $J(R/I_i)_n = JA_i = (JS_i)_{m_i}$, whence $(R/I_i)_n/J(RI_i)_n = A_i/JA_i = (S_i/JS_i)_{m_i} = (D_i)_{m_i}$, which is simple since D_i is a division ring. Thus R/I_i is a complete matrix ring over some local ring with T -nilpotent radical, i.e., R/I_i is a complete matrix ring over a Steinitzring, whence R/I_i is central perfect.

Suppose that $x \in I_1 \cap I_2 \cap \dots \cap I_k$. Then, if $\tilde{x} \in R_n$, with $\tilde{x}_{11} = x$ $\tilde{x}_{ij} = 0$ other

wise, it follows that \tilde{x} becomes 0 in $(R/I_i)_n = A_i$, i. e., $\tilde{x} = \tilde{x}_1 + \dots + \tilde{x}_k$ in $(S_1)_{m_1} \oplus \dots \oplus (S_k)_{m_k}$ with $\tilde{x}_i = 0$. But then $x = 0$ and $I_1 \cap \dots \cap I_k = 0$, so that $R = R/I_1 \oplus \dots \oplus R/I_k$ by the Chinese Remainder Theorem, whence R is a central perfect ring.

For the converse we note that if $R = A \oplus B$, then $R_n = A_n \oplus B_n$ and if $R = S_m$, then $R_n = S_{mn}$. Thus if R is central perfect then R_n is clearly central perfect since R is a perfect L. C. I. ring.

COROLLARY 1: *Suppose $R[G]$ is a central perfect ring. Then $R[G] = (A_1 \oplus \dots \oplus A_n)[G]$, where $A_i = (S_i)_{m_i}$, S_i a Steinitz ring. Thus $A_i[G] = (S_i[G])_{m_i}$ is central perfect and hence $S_i[G]$ is central perfect.*

Proof: By corollary 2, proposition 24, R is central perfect. The conclusion follows from propositions 23 and 27.

COROLLARY 2: *If $S_i[G]$ is central perfect for $i = 1, \dots, n$ and if $R = (S_1)_{m_1} \oplus \dots \oplus (S_n)_{m_n}$, then $R[G]$ is central perfect.*

Proof: If $S_i[G]$ is central perfect, then $(S_i)_{m_i}[G] = (S_i[G])_{m_i} = (A_1 \oplus \dots \oplus A_k)_{m_i} = (A_1)_{m_i} \oplus \dots \oplus (A_k)_{m_i}$ with $A_j = (T_j)_{n_j}$, where T_j is a Steinitz ring, whence $(A_j)_{m_j} = (T_j)_{n_j m_j}$ and $(S_i)_{m_i}[G]$ is also central perfect. Hence $R[G] = (S_1)_{m_1}[G] \oplus \dots \oplus (S_n)_{m_n}[G]$ is also central perfect.

COROLLARY 3: *If $R[G]$ is normal perfect, then $R[G] = (A_1 \oplus \dots \oplus A_n)[G]$, where $A_i = (S_i)_{m_i}$, S_i a normal Steinitz ring. Thus $A_i[G]$ is normal perfect and $S_i[G]$ is normal perfect.*

Proof: Since $R[G]$ is central perfect if it is normal perfect. The conclusion follows from propositions 17, 19 and corollary 1 above.

COROLLARY 4: *If $R[G] = (S_1)_{m_1}[G] \oplus \dots \oplus (S_n)_{m_n}[G]$, where $S_i[G]$ is a normal perfect ring for each i , then $R[G]$ is also a normal perfect ring.*

Proof: The corollary follows from propositions 17 and 19 as in the proof of corollary 2.

The corollaries to proposition 27 lead us to concentrate on the study of group rings over Steinitz rings in the next section.

6. Group rings over steinitz rings

If S is a Steinitz ring, then its characteristic is 0 or p^i for some prime p and integer i .

THEOREM 4: *If S is a Steinitz ring of characteristic $p^i > 0$ and if $S[G]$ is a central perfect ring, then G contains a normal subgroup H such that G/H*

is a p -group and such that $0(H)$, the order of H , is a unit in S .

Proof: Since $S[G]$ is central perfect, $S[G] = (S_1)_{m_1} \oplus \cdots \oplus (S_n)_{m_n}$, where the S_i are Steinitz rings.

Thus, let $1 = e_1 + \cdots + e_n$, where $\{e_1, \dots, e_n\}$ is a complete orthogonal set of central idempotents, with $e_i SG = (S_i)_{m_i}$

Hence, if N is thenorm $N: SG \rightarrow S$, then $N(e_i)$ is a central idempotent in S , and $N(e_i) = 1$ or $N(e_i) = 0$. Since $N(1) = 1$, $N(e_i) = 1$ for at least one i , say $N(e_1) = 1$. Since $N(e_1 e_i) = N(e_1) N(e_i) = 0$ for $i \neq 1$, it follows that $N(e_i) = 0$ if $i \neq 1$. Hence $(S_i)_{m_i} = e_i SG \subseteq \omega$; the augmentation ideal if $i \geq 1$ and

$$\omega = \omega_1 \oplus (S_2)_{m_2} \oplus \cdots \oplus (S_n)_{m_n}$$

with $\omega_1 = (I_1)_{m_1}$, $SG/\omega = S = (S_1)_{m_1}/\omega_1 = (S_1/I_1)_{m_1}$, whence $m_1 = 1$ and $I_1 = \omega_1$, since otherwise we would not obtain a Steinitz ring.

Let $H_i = \{g \in G \mid e_i g = g e_i = e_i\}$. The mapping $G \rightarrow SG$ given by $g \rightarrow e_i g$ is a homomorphism with kernel H_i , i. e., H_i is normal in G .

Furthermore, if $H_1 \cap \dots \cap H_n$ then $(e_1 + \cdots + e_n)g = g = 1$ so that $H_1 \cap \dots \cap H_n = \langle 1 \rangle$. Also, G/H_i is a group of units of $(S_i)_{m_i}$.

In order to prove the theorem we need only analyze the groups H_1 and G/H_1 .

We claim that G/H_1 is in fact a p -group. Indeed, for $g \in G$, $N(1-g) = 0$, $1-g \in \omega$, and $e_1 - e_1 g \in \omega_1$, i. e., in the mapping $S_1/\omega_1 \rightarrow S$, $e_1 - e_1 g \rightarrow 0$, whence since $e_1 \rightarrow 1$ we have $e_1 g \rightarrow 1$. The proof of the first assertion is thus complete if we prove the following:

LEMMA 7: Suppose that S is a Steinitz ring of characteristic $p^i > 0$, and suppose that U is a finite subgroup of $1+JS$. Then U is a p -group.

Proof: Let $T = \mathbf{Z}/(p^i)[U]$ be the ring generated by U over $\mathbf{Z}/(p^i)$ considered as a subring of S . We claim that T is itself a Steinitz ring. T is finite since it is a finite $\mathbf{Z}/(p^i)$ -module. Also, S/JS is a division ring, $A = T/T \cap JS \subseteq S/JS$ is a finite (non-commutative) integral domain contained in a division ring. Since A is algebraic over $\mathbf{Z}/(p)$, it follows that A is a finite division ring and hence a field. Therefore $T \cap JS$ is a maximal ideal of T and since $U \cap JS \subseteq JS$ is T -nilpotent, $T \cap JS = JT$, since otherwise $JT = T$, an obvious contradiction. Hence T is a Steinitz ring.

If A contains p^n elements, then since T/JT , $JT/(JT)^2$, ... are vector spaces over A , it follows that each of these contains p^{m_i} elements, where $m_i \equiv n \dim_A (JT)^i / (JT)^{i+1}$.

Hence JT contains p^m elements for some m and $1+JT$ is a p -group. Since $U \subseteq 1+JT$, it follows that U is a p -group as well.

Thus it follows that G/H_1 is a p -group.

Next we must show that $0(H_1)$ is a unit in S . We have $e_1 = \sum \alpha(g)g$ and for $h \in H_1$, $e_1 h = e_1 = \sum \alpha(g)gh$, whence $\alpha(gh^{-1}) = \alpha(g)$. Let $[kH_1] = \sum_{h \in H_1} kh$. Then, if g_1, \dots, g_t is a set of coset representatives:

$$\begin{aligned} e_1 &= \sum_{i=1}^t \alpha_i [g_i H_1], \text{ and} \\ e_1^2 &= \sum_{i,j=1}^t \alpha_i \alpha_j [g_i H_1] [g_j H_1] \\ &= \sum_{i,j=1}^t \alpha_i \alpha_j [g_i g_j H_1] 0(H_1) \\ &= \sum (\sum_{g_i g_j H = g_k H} \alpha_i \alpha_j 0(H_1)) [g_k H_1] = e_1 \end{aligned}$$

and $\alpha_k = \sum_{g_i g_j H = g_k H} \alpha_i \alpha_j 0(H_1)$.

Also, $N(e_1) = 1 = 0(H_1) (\sum_k \sum_{g_i g_j H = g_k H} \alpha_i \alpha_j)$, whence $0(H_1)$ is a unit in S as asserted. If we let $H = H_1$, then the theorem follows.

Thus, e.g., if $S = \mathbb{Z}/(3)$ and $G = S_3$, then $S[G]$ is perfect but not central perfect, since by the theorem G would have to contain a normal subgroup H with G/H a 3-group and $0(H)$ a unit in $\mathbb{Z}/(3)$. The theorem provides us with another easy method of constructing perfect rings which are not central perfect.

From proposition 22 it follows that if $H = \langle 1 \rangle$, i.e., $G/H = G$ is a (finite) p -group, then $S[G]$ is central perfect, so that there are conditions on G (or on G/H and H) which will imply the converse in all cases.

We derive another such condition for normal Steinitz rings.

THEOREM 5: *Suppose that S is a normal Steinitz ring and suppose that H is a finite group such that $0(H)$ is a unit in S . Then $S[H]$ is a central perfect ring.*

Proof: We claim that $S[H]/JS[H] = (S/JS)[H]$. Indeed, let us map $S[H] \rightarrow (S/JS)[H]$ by letting $\alpha = \sum \alpha(g)g \rightarrow \sum (\alpha(g) + JS)g$. This mapping is an epimorphism with kernel $(JS)[H]$.

Under the hypotheses of the theorem we may replace the notion of field by the notion of Steinitz ring in Passman's Lemma 7.2.2, Theorem 7.2.7 and Theorem 7.2.10 [cf. 7, pp. 274-275, 278-279]. As a result, it follows for the subgroup $\langle 1 \rangle$ that $JS[H] = (JS\langle 1 \rangle)S[H] = (JS)S[H] = (JS)[H]$ whence the claim follows from the fundamental theorem of homomorphisms. Since (S/JS) is a division ring, and since $0[H]$ is a unit in (S/JS) , it follows that $(S/JS)[H]$ is a semi-simple Artinian ring by Connell's generalization of Maschke's theorem.

In particular, $A = (S/JS)[H]$ contains a complete orthogonal set of central idempotents $\bar{I} = \bar{e}_1 + \dots + \bar{e}_k$, where the \bar{e}_i are minimal, i.e., the rings $\bar{e}_i A \bar{e}_i$ are complete matrix rings over division rings.

If $\bar{U} \in CA$, then $\bar{U} = \sum (\alpha(g) + JS)g$, $\alpha(g) + JS \in C(S/JS)$. Since S is a normal ring, $(CS + JS)/JS = C(S/JS)$, i.e., there is an element $\beta(g) \in CS$

such that $\beta(g) + JS = \alpha(g) + JS$.

Since $\alpha(g) + JS$ is a class function, i. e., $\alpha(xgx^{-1}) + JS = \alpha(g) + JS$, we may select $\beta(xgx^{-1}) = \beta(g)$, and thus under these circumstances $U = \sum \beta(g)g$ is an element in the center of $S[H]$ such that $U + JS[H] = \bar{U}$.

Since $S[H]$ is perfect, $JS[H]$ is T -nilpotent and thus we may construct a central idempotent e_1 mapping to a central idempotent \bar{e}_i according to the standard recipe:

$$\begin{aligned} e_i &= u + x(1 - 2u), \quad x = 1/2(1 - (1 + 4n)^{1/2}), \\ n &= u^2 - u, \quad u \rightarrow \bar{e}_i \text{ and } u \in CS[H]. \end{aligned}$$

Since u is in the center and since e_i is in fact a linear combination of powers of u , e_i is itself in $CS[H]$.

It follows that if we construct central idempotents e_1, \dots, e_k , then $e_i e_j$ is a central idempotent which is in $JS[H]$ if $i \neq j$, whence $e_i e_j = 0$, i. e., the set $\{e_1, \dots, e_k\}$ is also an orthogonal set of central idempotents.

Similarly, $e_1 + \dots + e_k = \mu$ is a central idempotent with $1 - \mu \in JS[H]$, i. e., $\mu = 1$ and the set is complete.

By the definition it follows that $S[H]$ is central perfect.

COROLLARY 1: *If S is a normal Steinitz ring of characteristic 0, then $S[G]$ is central perfect if and only if G is finite.*

Proof: If S is a Steinitz ring of characteristic 0, then S contains the rational numbers and hence if G is finite, $0(G)$ is a unit in S . The conclusion follows from theorem 5.

The phrasing of theorem 5 does not allow us to replace the word "normal" by the word "central" since in the proof we make use of the normality of S only once, but in an essential manner.

With theorem 5 we can handle the situation where we have a split extension $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ with G/H a finite p -group and $0(H)$ a unit in S , a normal Steinitz ring of characteristic $p_i > 0$. Indeed, in this case it follows that $S[G] = S[H][G/H]$, where $S[H]$ is a central perfect ring, i. e., $S[H] = (T_1)_{m_1} \oplus \dots \oplus (T_n)_{m_n}$, so that by proposition 23, $S[H][G/H] = (T_1[G/H])_{m_1} \oplus \dots \oplus (T_n[G/H])_{m_n}$, where $T_j[G/H]$ is a Steinitz ring since T_j has characteristic $p^{j(i)} > 0$ and since G/H is a finite p -group. Thus we have the following.

COROLLARY 2: *If S is a normal Steinitz ring of characteristic $p^i > 0$, and if $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ is a split extension with G/H a finite p -group and $0(H)$ a unit in S , then $S[G]$ is central perfect.*

To continue with our discussion consider the following propositions as adapted from Passman's book with the word "field" replaced by "Steinitz ring".

LEMMA 6.1.5. Let R be a ring which contains a set of elements $\{e_{ij} | i, j = 1, \dots, n\}$ satisfying

$$e_{ij}e_{ab} = 0 \text{ for } j \neq a, \quad e_{ij}e_{ab} = e_{ib} \text{ for } j = a \text{ and } 1 = e_{11} + e_{22} + \dots + e_{nn}.$$

If A is the centralizer in R of all these elements, then $R \cong M_n(A)$ and $A = e_{11}Re_{11}$.

LEMMA 6.1.6. Let R be a ring and let $1 = e_1 + \dots + e_n$ be a decomposition of one into a sum of orthogonal idempotents. Let G be a subgroup of units of R and assume G permutes the set $\{e_{11}, \dots, e_{nn}\}$ transitively by conjugation. Then $R \cong M_n(A)$, where A is the ring $A = e_1Re_1$.

LEMMA 6.1.7. Let G be a finite group and let $H \triangleleft G$. Suppose $\{e_1, \dots, e_n\}$ is a G -orbit of centrally primitive idempotents of $S[H]$. Then $e = e_1 + \dots + e_n$ is a central idempotent of $S[G]$ and $eSG \cong M_n(e_1SG_1)$ where $G_1 \supseteq H$ is the centralizer of e_1 in G .

LEMMA 6.1.8. Let G be a finite group and let $H \triangleleft G$. Suppose $e \in S[H]$ is a central idempotent of $S[G]$ with $eS[H] \cong M_m(S)$. Then $eS[G] \cong M_m(S^t[G/H])$, where $S^t[G/H]$ is some twisted group ring of G/H .

THEOREM 6.1.9. Let G be a finite group and let $H \triangleleft G$. Suppose that $\{e_1, \dots, e_n\}$ is a G -orbit of centrally primitive idempotents of $S[H]$ with $e_1S[G] \cong M_m(S)$. Then $e = e_1 + \dots + e_n$ is a central idempotent of $S[G]$ and

$$eS[G] \cong M_{mn}(S^t[G_1/H]),$$

where $G_1 \supseteq H$ is the centralizer of e_1 in G and where $S^t[G_1/H]$ is some twisted group ring of G_1/H .

Suppose that S is a normal Steinitz ring of characteristic $p^i > 0$. Then let G be a finite group and suppose $H \triangleleft G$ has the property that G/H is a p -group and $0(H)$ is a unit in S . It follows from theorem that $S[H]$ is a central perfect ring, say

$$S[H] = (S_1)_{m_1} + \dots + (S_n)_{m_n}$$

with $1 = e_1 + \dots + e_n$, where $e_iS[H] = (S_i)_{n_i}$ and where e_i is a centrally primitive idempotent for each i .

Now the group G acts on $\{e_1, \dots, e_k\}$ by conjugation and if G_i is the centralizer of e_i in G , then the fact that e_i and e_j are conjugates implies that G_i and G_j are conjugates as well.

Since $H \subseteq G_i \subseteq G$, it follows that $H \triangleleft G_i$ and that G_i/H is a p -group. Use lemma 6.1.7 to decompose 1 as a sum of centrally primitive idempotents of $S[G]$, say $1 = f_1 + \dots + f_l$ where $f_1 = e = e_1 + \dots + e_n$ and where $\{e_1, \dots, e_n\}$ is without loss of generality the G -orbit of e_1 , with the rest of the f_i 's constructed in a similar fashion. Then

$$S[G] = f_1 S[G] + \dots + f_t S[G] = M_{n_1}(e_1 S[G_1]) + \dots + M_{n_t}(e_t S[G_t])$$

where the G_i are as defined above.

If $S[G]$ is a central perfect ring, then by propositions 12 and 13 the rings $M_n(e_j S[G_j])$ are themselves central perfect and by proposition 27, $e_j S[G_j]$ is a central perfect ring.

Conversely, if $e_j S[G_j]$ is a central perfect ring in all cases, then $S[G]$ is itself a central perfect ring.

If the groups G_i are such that the rings $S[G_i]$ are all central, e.g., if $1 \rightarrow H \rightarrow G_i \rightarrow G_i/H \rightarrow 1$ is a split exact sequence for all i , then since $e_j S[G_j]$ is a direct summand of a central perfect ring, for each j , it follows that the rings $e_j S[G_j]$ are central perfect rings by propositions 12 and 13.

EXAMPLE 4: If $R = \{0, 1\}$ is the field with two elements and if $G = S_3$, then $R[G] = R[S_3] = R_2 \oplus \tau_2(R)$, which is seen to be a normal perfect ring. Thus even in the situation where we're dealing with group algebras, Steinitz rings enter in an unavoidable manner.

We note that this example is an example in support of a converse of theorem 4 which we state in the form of a conjecture.

CONJECTURE: If S is a (normal) Steinitz ring of characteristic $p^i > 0$ and if $H \triangleleft G$ is a finite p' -group such that G/H is a finite p -group, then $S[G]$ is a central (normal) perfect ring.

The representation of $R[G]$ given in example 4 was arrived at using a program, a computer and some hand computations involving the idempotents of $R[G]$. It is our purpose to discuss these and similar computations elsewhere.

From proposition 22 and following, it seems that at least in the case of perfect rings the classification problems listed at the beginning of the section on group rings are very much related. At present we are not in a position to give complete solutions due to the necessity of assumptions concerning normality as in theorem 5, and problems with the arithmetic of twisted group rings over Steinitz rings. It is our conjecture that these problems are only technical and not intrinsic, i.e., the converse to theorem 4 is true.

If $R[G]$ is perfect but not central perfect, then using a standard normal decomposition we may write $R[G] = R[G]^* \oplus_\phi M$. It would be interesting to relate the structure of M to that of R and G in general. The characterization given in the first section along with a program for computing idempotents provides a (slow) method for determining M . As a problem in computation, improved methods would always be interesting in any search for examples or counterexamples to conjectures, including ones made above.

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