J. Korean Math. Soc.
Vol. 17, No. 1, 1980

SOME MAXIMAL EQUIDISTANT PERMUTATION ARRAYS

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1. Introduction

An equidistant permutation array (*EPA*) is a $v\times r$ array defined on a symbol set V such that

(1) every row is a permutation of the symbol set V

(2) every pair of distinct rows has precisely λ common colume entries. We denote such an array by $A(r, \lambda; v)$.

An *EPA*, $A=(r, \lambda; v)$ is said to be maximal if it is impossible to add a $(v+1)$st row to A such that the resulting array is an $A(r, \lambda; v+1)$. A maximal $A(r, \lambda; v)$ is denoted by $\bar{A}(r, \lambda; v)$. Define

$$R(r, \lambda) = \max\{v: \text{ there exists an } A(r, \lambda; v)\}$$

and

$$\bar{R}(r, \lambda) = \min\{v: \text{there exists an } \bar{A}(r, \lambda; v)\}.$$

There are a number of results on the function $R(r, \lambda)$. Some of these results can be found in [3], [5] and [7]. In this paper, we are interested in maximal *EPA*s. We require several more definitions.

A *generalized Room square* (GRS) is an $r\times r$ array defined on a symbol set V of cardinality v such that

(1) every cell contains a (possibly empty) subset of V,

(2) every element of V is contained in every row and column of the array precisely once and

(3) every pair of distinct elements of V is contained in λ of the cells.

Such an array will be denoted by $S(r, \lambda; v)$.

An (r, λ)-design D is a collection B of subsets (blocks) taken from a finite set V of elements (varieties) such that

(1′) every element of V is contained in precisely r blocks of D

(2′) every pair of distinct elements of V is contained in exactly λ blocks of D.

If each block of D has cardinality k, D has v varieties and 6 blocks,

Received Jan. 8, 1980

*) Research supported by NSERC under grant #A9258

then D is called a balanced incomplete block design and is denoted (v, b, r, k, λ)-*BIBD*.

In section 2, (r, λ)-designs are used to generalize a construction for maximal *EPA*s which was given in [6]. In section 3, we show that this construction applied to *BIBD*s has been shown [6] to give maximal *EPA*s.

2. *EPA*s and $(r, 1)$-designs.

Let a be a permutation acting on the set $N=\{1, 2, 3, \ldots, n\}$. Define

$$E(a)=\{i\in N : a(i)\neq i\}.$$

For any $A\subseteq S_n$, $E(A)=\{E(a) : a\in A\}$. Let $L=\{l_1, l_2, \ldots, l_t\}$ be a set of positive integers. $A(n, L)$ is a set of permutations such that for all $a, b\in A$, $(a\neq b)$, $n-|E(a^{-1}b)|\in L$. In the case where $|L|=1$, $A(n, L)$ is just an *EPA*.

LEMMA 2.1. *Let $A=A(n, L)$ such that*

(i) $|E(a)\cap E(b)|\leq 1$ *for all* $a, b\in A$, $a\neq b$.

Then:

a) $|E(A)|=|A|$;

b) $E(a^{-1}b)=E(a)\cup E(b)$ *for all* $a, b\in A$, $a\neq b$ *and hence* $n-|E_1\cup E_2|\in L$ *for all* $E_1, E_2\in E(A)$, $E_1\neq E_2$;

c) *A is a proper subset of some $A(n, L)$ with property* (i) *iff there exists* $E'\subset\{1, 2, \ldots, n\}$ *such that* $|E'|\neq 1$, $E'\notin E(A)$ *and* $n-|E'\cup E|\in L$ *for any* $E\in E(A)$.

Proof. a) We have, of course, $|E(A)|\leq|A|$ from definition of $E(A)$. Suppose $|E(A)|<|A|$. Hence $E(a)=E(b)$ for some $a, b\in A$, $a\neq b$. We obtain $E(a)\cap E(b)=E(a)=E(b)$; condition (i) implies $E(a)\leq 1, E(b)\leq 1$. But neither $E(a)=1$ nor $E(b)=1$ is possible. We have $|E(a)|=|E(b)|=0$, i.e., both a, b are the identity permutation. This contradicts the supposition $a\neq b$.

b) For any $a, b\in S_n$ we have $E(a) \ \nabla E(b)\subseteq E(a^{-1}b)\subseteq E(a)\cup E(b)$ where $E(a)\nabla E(b)=(E(a)\cup E(b))-(E(a)\cap E(b))$ is the symmetric difference of the sets $E(a), E(b)$. In case $|E(a)\cap E(b)|=0$ we have $E(a^{-1}b)=E(a)\cup E(b)$ immediately. Suppose now that $|E(a)\cap E(b)|=1$ and $E(a)\cap E(b)=\{p\}$ for some $p\in\{1, \ldots, n\}$. We have $a(p)\neq p\neq b(p)$. The case $a(p)=b(p)$ will imply $a(a(p))\neq a(p)$, $b(a(p))\neq a(p)$, that is, $a(p)\in E(a)\cap E(b)$ and $|E(a)\cap E(b)|>1$ which contradicts to condition (i). Hence, $a(p)\neq b(p)$ and $E(a^{-1}b)=E(a)\cup E(b)$.

c) This follows from *a*) and *b*).

In the next theorem we give a construction for a class of $A(n, L)$.

THEOREM 2.1. *If there exists an $(r, 1)$-design D with n blocks and v varieties, then there exists an $A(n, n-2r+1; v)$.*

Proof. Let $T=\{B_1, B_2, \ldots, B_n\}$ be the blocks of D. For each variety $x\in D$, let f_x be any permutation acting on T such that $E(f_x)=\{B : x\in B\}$. We now show that $A=\{f_x : x\in D\}$ is an $A(n, l; v)$. It is clear that A satisfies property (i) of Lemma 2.1. Hence, from (b) of Lemma 2.1, $E(a^{-1}b)=E(a)\cup E(b)$ for all $a, b\in A$ $(a\neq b)$. But $E(a)=r$, for all $a\in A$ and, thus, $E(a^{-1}b)=2r-1$. This completes the proof.

This result generalizes a construction given in [6].

COROLLARY 2.1. *Let $A=A(n, l)$ be the permutation array corresponding to a given $(r, 1)$-design D. If G is the GRS corresponding to A, then the blocks of G having cardinality greater than one are precisely the blocks of the complement to D.*

Proof. A block B of G of cardinality $k\geq 2$ corresponds to a set $\{a_1, a_2, \ldots a_k\}$ permutations of A with the property that $a_i(b)=a$ for all i, $1\leq i\leq k$ and fixed elements a and b. Since $k\geq 2$, and $t(a_i, a_j)=0$ for all i, j $(i\neq j)$, then $a\notin E(a_i)$, $1\leq i\leq k$. Hence, the elements of D associated with $a_1, a_2, \ldots, a_k$ do not occur in the block B' of D associated with b. Hence, the elements of D associated wth $a_1, a_2, \ldots, a_k$ form the block $\overline{B}'$. This completes the proof.

We remark that the same procedure applied to an (r, λ)-design will produce an $A(n, L)$ where $L=\{n-2r+\lambda+t : 0\leq t\leq\lambda-1\}$. This result follows from the fact that for any two permutations $a, b (a\neq b)$ $|E(a^{-1}b)|=|E(a)|+|E(b)|-|E(a)\cap E(b)|-t(a, b)$ where $t(a, b)=|\{i\in E(a)\cap E(b) : a(i)\}|$ and the fact that $0\leq t(a, b)\leq\lambda-1$.

Theorem 2.1 shows that any $(r, 1)$-design with n blocks can be used to produce an $A(n, n-2r+1)$. Given an $A(n, l)$, when can we produce an $(r, 1)$-design. An $A(n, l)$is said to have property (i′) if $|E(a)|=r=\dfrac{n+1-l}{2}$, for all $a\in A$ and $|E(a)\cap E(b)|=1$ for all $a, b\in A$ $(a\neq b)$. It is easily seen that there exists an $(r, 1)$-design with n blocks if and onlyif there exists an $A(n, n-2r+1)$ having property (i′).

LEMMA 2.2. *Let $A=A(n, l)$ have property* (i′) *and $A\cup\{a'\}$ be an $A(n, l)$ for some $a'\in S_n\setminus A$. Then*

a) $|E(a')|\geq r-1$.

b) $|E(a')|=r-1$ *iff* $E(a')\cap E(a)=\phi$, *for all* $a\in A$.

$|E(a')|=r$ *iff* $|E(a')\cap E(a)|=1$, *for all* $a\in A$.

$|E(a')|=r+1$ *iff* $|E(a')\cap E(a)|=2$, *and* $t(a, a')=0$ *for all* $a\in A$.

c) $|E(a')| \geq r+2$ *implies* $|E(a') \cap E(a)| \geq 2$, *and*
$|E(a') \cap E(a)| + t(a, a') = |E(a')| - r + 1 \geq 3$ *for all* $a \in A$.

Proof. Recall $|E(a^{-1}a')| = |E(a)| + |E(a')| - |E(a) \cap E(a')| - t(a, a')$. Since $|E(a)| = r$ for all $a \in A$ and $|E(a^{-1}b)| = n-l$ for all $a, b \in A (a \neq b)$ then $|E(a')| = r - 1 + |E(a) \cap E(a')| + t(a, a')$. (a) now follows. (b) follows from the above and Lemma 1 using the fact $|E(a) \cup E(a')| \leq 1$ implies $t(a, a') = 0$. This completes the proof.

Recall that $A = A(n, l)$ is maximal and denote it by $\bar{A}(n, l)$ if $A \cup \{a'\} \neq A(n, l)$ for any $a \in S_n \setminus A$. An $(r, 1)$-design D with V varieties and b blocks is extendible to an $(r, 1)$-design D' with $v+1$ varieties and b blocks if D is isomorphic to a restriction of D'. This is a special case of the extension given implicitly in c) of Lemma 1. An $A = A(n, l)$ with property (i$'$) is strongly extendible if there exists an element $a \in S_n \setminus A$ such that $A \cup \{a\}$ is an $A(n, l)$ with property (i$'$).

THEOREM 2. 2. *An* $A(n, l)$ *with property* (i$'$) *is strongly extendible iff the corresponding* $(r, 1)$*-design is extendible.*

COROLLARY 2. 2. *An* $A(n, l;\ v)$ *with property* (i$'$) *is extendible to an* $A(n, l; n)$ *with property* (i$'$) *if* $v > \left\lceil \frac{n-l-1}{2} \right\rceil^2$.

Proof. This follows from a result on extendible $(r, 1)$-designs which can be found in [8].

We conclude this section with the following useful lemmas on *EPA*s $A = A(n, l, v)$ with $|A| = v \geq 2$. Suppose that $|E(a)| = r$ for any $a \in A$. It is evident that $r \geq 2$.

LEMMA 2. 3 *Suppose that* $|E(a)| = 2$ *for any* $a \in A$. *Then either*

a) $l = n-4$, $E(a) \cap E(b) = \phi$ *for any* $a, b \in A$, $a \neq b$; or

b) $l = n-3$, $|\bigcap_{a \in A} E(a)| = 1$; or

c) $l = n-3$, $A = \{(i, j), (i, k), (j, k)\}$ *for some* $1 \leq i < j < k \leq n$.

Proof. A consists of transpositions, i. e., cycles of length 2. The case a) corresponds to the possibility that every pair of these distinct cycles has no element of $\{1, 2, \ldots, n\}$ in common. The only possibility is that every pair of these cycles has exactly one element in common. For this case Lemma (3. 4) of [4] gives only two possibilities corresponding to cases b) and c).

REMARK. Using known equalities ([3]),

$$\max|A(n, n-3)|=n-1, \quad \max|A(n, \ n-4)|=[n/2]$$

we can show easily that

in the case a) of Lemma 2.1, $A=\bar{A}(n,n-4)$ iff $v=[n/2]$;

in the case b) of Lemma 2.1, $A=\bar{A}(n,n-3)$ iff $v=n-1$.

Also, $A=\bar{A}(n,n-3)$ in the case c) of Lemma 2.1.

LEMMA 2.4. *Let* $A=A(n,l;v)$. *Then* $l\neq n-1$ *and* $l\leq n-3$ *for* $|A|>2$.

Proof. It follows trivially from Lemma 2.1. In fact, suppose $l=n-2$ and a_1, a_2 and a_3 are three distinct elements of A. Then $\{a_1^{-1}a_2, a_1^{-1}a_3\}$ is an $A(n, \ n-2; \ 2)$ with $|E(a_1^{-1}a_2)|=|A(a_1^{-1}a_3)|=2$, which is an impossibility.

LEMMA 2.5. *Let* $A=A(n,l;v)$, $v>2$ *and let* $a'\in A$.
Denote $A'=\{b^{-1}a' : b\in A, b\neq a'\}$; *then*

a) $A'=(n,l;v-1)$ *with*

b) $|E(a)|=n-l$ *for any* $a\in A'$ *and*

c) $|E(a)\cap E(b)|\geq\max\left[\frac{n-l}{2}, 2\right]$ *for any* $a, b\in A'$, $a\neq b$.

Proof. a) and b) follows from the fact that Hamming distance $|E(a^{-1}b)|$ on S_n is invariant of translation, i.e.,

$$|E(a^{-1}b)|=|E((ac)^{-1}(bc))|=|E((ca)^{-1}(cd))|.$$

Let $a, b\in A'$, $a\neq b$.

$$n-l=|E(a^{-1}b)|\geq|E(a)\nabla E(b)|=|(E(a)\cup E(b))-E(a)\cap E(b)|$$
$$=|E(a)|+|E(b)|-2|E(a)\cap E(b)|=2(n-l)-2|E(a)\cap E(b)|.$$

Hence, $|E(a)\cap E(b)|\geq(n-l)/2$. Moreover, $(n-l)/2>1$ (because $l\leq n-3$) from Lemma 2.1 and $|E(a)\cap E(a)|$ is an integer; so c) is proved.

3. Maximal *EPA*s

In this section we find several classes of maximal *EPA*s.

Let $LS(n+l, \ l; n)$ be an $A(n+l, l; n)$ obtained from a latin square by adjoining l fixed points.

THEOREM 3.1. *For each positive integer* n *and* l, $LS(n+l, \ l; \ n)$ *is an* $\bar{A}(n+l, \ l; n)$

Proof. Let G be the *GRS* associated with *LS*. The nonemply cells of G contain blocks of size 1 and n. Without loss of generality, assume that the blocks of size n occur in $l\times l$ subarray S and S occurs in the upper left corner of G and L is the latin square subarray in the lower right corner. Suppose G is extendible by adjoining a new element x. If x is contained in all blocks of size n, x cannot occur in any cell of L. Hence, x cannot occur in any row or column of G which contains L. Suppose there is some block B of size n which does not contain x. x must occur once in the row of G which contains B. This row does not contain a row of L. Since x must occur with every element l times, x must occur in each row and column of L. This is impossible. Hence, G is not extendible and LS is maximal.

This theorem implies that $\bar{R}(n, l) \leq n-l$. We conjecture that $\bar{R}(n, l) = n-l$.
Another class of maximal EPAs is given in the next theorem.

THEOREM 3.2. *If there exists* $a(v, b, r, k, l)$-*BIBD*, $(k \geq 3)$, *then there exists an* $\bar{A}(b, b-2r+1;\ v)$.

A proof of this theorem can be found in [9]. In this special case, where $v=b$, the result stated in Theorem 3.2 was obtained in [6].

4. $A(n, l; n)$:

We now consider a special class of maximal EPAs. An EPA $A(n, l; v)$ is called square if $n=v$. We are interested in $\bar{A}=A(n, l; n)$. For $l=0$, a latin square provides an example of an $\bar{A}(n, 0; n)$ for any n. *For* $l=q^2-q$, $n=q^2+q+1$ and, if there exists a finite projective projective plane, then there exists ([6]) an $\bar{A}(q^2+q+1,\ q^2-q;\ q^2+q+1)$.

It is shown ([1]) that

$$|A(n, l)| \leq \max\{l+2,\ (n-l)^2+(n-l)+1\}.$$

If $(n-l)^2+(n-l)+1 \leq l$, then $|A(n, l)| \leq l+2$ and, hence, from lemma 2.4 $|A(n, l)| < n$ and the square cannot exist. This will occur provided $l \geq n-1-\sqrt{n+2}$. Let $f(n)$ be the maximum value of l such that an $A(n, l; n)$ exists.

THEOREM 4.1.

a) $f(n) < n-1-\sqrt{n+2}$

b) *Let* $n=q^2+q+1$ *for* q *a prime power* p^a. *Then*

$$n-1-2\sqrt{n-1} < q^2-q \leq f(n) \leq q^2-1.$$

c) $f(7)=2$.

Proof. Part (a) follows from the above. The lower bound in (b) follows from the results of [1]. The upper bound follows from (a), and the fact that $q^2+q-\sqrt{q^2+q+3} < q^2+q-\sqrt{q^2} = q^2$.

In the case of $n=7$, $f(7)<3$ or $f(7)\leq 2$. From (b) of the theorem, $f(7)\geq 2$. Hence, $f(7)=2$. This completes the proof.

We conjecture that the lower bound in (b) is the exact value of $f(n)$. Since there exists an $A(6,1;7)$ ([3]), there exists an $A(7,2;7)$ which is not obtained from a finite projective plane using theorem 3. 1. Thus, the extremal case for $f(2)$ is not unique. We remark, however, that the $A(7, 2; 7)$ just given is not maximal. Another example of a square having the parameters of a square constructed in [6] but not obtainable by this construction is an $A(43,30;43)$. To construct such an array by the result of [6] requires a finite projective plane of order 6. This array can be constructed as follows. By a construction given in [5], there exists an $A(2q-1,\ q-3;\ q(q-2))$ If we take $q=11$, we obtain an $A(21,8;99)$. Deleting 56 permutations and adding 22 fixed points, produces the required array.

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