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RS-COMPACT SPACES

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1. Introduction and preliminaries

In this paper, we define RS-compact and weak RS-compact spaces which are strong forms of Quasi H-closed (A space X is Quasi H-closed if every open cover has a finite subfamily whose closures cover X) [1], nearly compact (A space X is nearly-compact if every open cover has a finite subfamily, the interiors of the closures of which cover X) [3] and S-closed (A space X is S-closed if every semiopen cover has a finite subfamily whose closures cover X) [7]. We show relationships among the spaces memtioned above and we investigate properties of RS-compact, weak RS-compact spaces. We also discuss the necessary and sufficient conditions for the images and the inverse images of RS-compact spaces to be RS-compact under almost-continuous [6] open mappings.

In a topological space X a set A is called *regular open* (*regular closed*) [2] if $A=\mathrm{Int}(\mathrm{Cl}(A))$ (resp. $A=\mathrm{Cl}(\mathrm{Int}(A))$). A set A of X is called *semiopen* [4] if there is an open set U such that $U\subset A\subset \mathrm{Cl}(U)$. A set A of X is called *regular semiopen* [1] if there is a regular open set V such that $V\subset A\subset \mathrm{Cl}(V)$. It is obvious that $\mathrm{Int}(A)=\mathrm{Int}(\mathrm{Cl}(A))$ for every regular semiopen A.

As is to be shown, the concept of RS-compact (weakRS-compact) is induced from the above definitions.

DEFINITION. A topological space X is *RS-compact* (*weak RS-compact*) if every regular semiopen cover has a finite subfamily whose interiors cover X (resp. if every regular semiopen cover has a finite subcover).

An easy consequence of these definitions is

THEOREM 1. 1. *The following implications hold and none of these implications can, in general, be reversed:*

$$\begin{tikzcd}
 & \text{nearly-compact} \arrow[d, "(3)"] & & \\
\text{compact} \arrow[ur, "(1)"] \arrow[r, "(2)"] & \text{Quasi } H\text{-closed} & \text{weak } RS\text{-compact} \arrow[ul, "(5)"'] \arrow[l, "(6)"'] \arrow[dl, "(7)"] & RS\text{-compact} \arrow[l, "(8)"'] \\
 & S\text{-closed} \arrow[u, "(4)"] & &
\end{tikzcd}$$

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Proof. The proofs of (1), (2), (4), (5) and (8) are shown evidently from each definitions and [1], [3] and [7].

First, we prove (5) and (6). Let $\{U_\alpha | \alpha \in I\}$ be an open cover of X. Then $\{\mathrm{Int}(\mathrm{Cl}(U_\alpha)) | \alpha \in I\}$ is a regular open cover of X. Since X is weak RS-compact, there exists a finite subfamily such that $\bigcup_{i=1}^{n} \mathrm{Int}\,(\mathrm{Cl}(U_{\alpha_i})) = X$. Therefore, $\bigcup_{i=1}^{n} \mathrm{Int}(\mathrm{Cl}(U_{\alpha_i})) \subset \bigcup_{i=1}^{n} \mathrm{Cl}(U_{\alpha_i}) = X$, and consequently X is nearly-compact and Quasi H-closed. Finally, we prove (7). Let $\{V_\alpha | \alpha \in I\}$ be a semiopen cover of X. Then $\{V_\alpha \cup \mathrm{Int}(\mathrm{Cl}(V_\alpha)) | \alpha \in I\}$ is a regular semiopen cover of X. By hypothesis, there exists a finite subfamily such that $\bigcup_{i=1}^{n}(V_{\alpha_i} \cup \mathrm{Int}(\mathrm{Cl}(V_{\alpha_i}))) = X$. Therefore, $\bigcap_{i=1}^{n}(V_{\alpha_i} \cup \mathrm{Int}(\mathrm{Cl}(V_{\alpha_i}))) \subset \bigcup_{i=1}^{n} \mathrm{Cl}(V_{\alpha_i}) = X$, X is S-closed.

Our knowledge of the properties of an extremally disconnected spaces [8] immediately gives the next theorem whose proofs are omitted.

THEOREM 1. 2. *In an extremally disconnected space X, the following are equivalent.*

(a) *X is nearly-compact.*

(b) *X is Quasi H-closed.*

(c) *X is S-closed.*

(d) *X is RS-compact.*

(e) *X is weak RS-compact.*

2. Characterizations of RS-compact and weak RS-compact spaces

In this section we discuss some properties of RS-compact spaces, weak RS-compact spaces and properties of the previous two spaces that arise from a certain class of mappings.

THEOREM 2. 1. *If a space X is RS-compact, then for every family $\{F_\alpha | \alpha \in I\}$ of regular semiclosed* (i. e., *each F_α is the complement of a regular semiopen set) sets in X satisfying $\bigcap_\alpha F_\alpha = \phi$ there is finite subfamily $F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_n}$ with $\bigcap_{i=1}^{n} F_{\alpha_i} = \phi$.*

Proof. Let $\{F_\alpha | \alpha \in I\}$ be a family regular semiclosed sets in X satisfying $\bigcap_\alpha F_\alpha = \phi$. Then $\{X - F_\alpha | \alpha \in I\}$ is a regular semiopen cover of X. Since X is RS-compact, there exists a finite subfamily such that $\bigcup_{i=1}^{n} \mathrm{Int}(X - F_{\alpha_i}) = X \subset \bigcup_{i=1}^{n}(X - F_{\alpha_i})$. Therefore, $\bigcup_{i=1}^{n} F_{\alpha_i} = \phi$.

The finite intersection property of regular closed sets holds, since a regular closed set is regular semiclosed. Butthe converse of Theorem 2. 1 is not true. From next theorem, we see that weak RS-compact is the necessary and sufficient condition for the finite intersection property of regular semiclosed.

THEOREM 2. 2, *In a space X the following are equivalent.*

(a) *X is weak RS-compact.*

(b) *For every family $\{F_\alpha \mid \alpha \in I\}$ of regular semiclosed sets in X satisfying $\bigcap_\alpha F_\alpha = \phi$, there is a finite subfamily $F_{\alpha_1}, F_{\alpha_2}, \dots, F_{\alpha_n}$ such that $\bigcap_{i=1}^n F_{\alpha_i} = \phi$.*

Proof. (a)$\Rightarrow$(b). Let $\{F_\alpha \mid \alpha \in I\}$ be a family regular semiclosed sets in satisfying $\bigcap_\alpha F_\alpha = \phi$. Then $\{X - F_\alpha \mid \alpha \in I\}$ is a regular semiopen cover of X. Since X is weak RS-compact, there exists a finite subfamily such that $X = \bigcup_{i=1}^n (X - F_{\alpha_i})$. Hence $\bigcap_{i=1}^n F_{\alpha_i} = \phi$.

(b)$\Rightarrow$(a). Let $\{V_\alpha \mid \alpha \in I\}$ be a regular semiopen cover of X. Then $\{X - V_\alpha \mid \alpha \in I\}$ is a family of regularsemiclosed sets satisfying $\bigcap_\alpha (X - V_\alpha) = \phi$. By hypothesis, there exists a finite subfamily such that $\bigcap_{i=1}^n V_{\alpha_i}$. Therefore, $X = U_{i=1}^n V_{\alpha_i}$ and X is weak RS-compact.

THEOREM 2. 3. *A space X is RS-compact if and only if every regular closed subset C of X is RS-compact.*

Proof. First, let C be a regular closed subset of X and let $\{V_\alpha \mid U_\alpha \cap C = V_\alpha,\ U_\alpha$ regular semiopen in X for each $\alpha \in I\}$ be a family regular semiopens of C satisfying $\bigcup_\alpha V_\alpha = C$. Then $\{U_\alpha \cup (X - C) \mid \alpha \in I\}$ is a regular semiopen cover of X. Since X is RS-compact, there exists a finite subfamily such that $\bigcup_{i=1}^n \mathrm{Int}(U_{\alpha_i} \cup (X - C)) = \bigcup_{i=1}^n \mathrm{Int}(U_{\alpha_i}) \cup (X - C) = X$. Therefore, $C = \bigcup_{i=1}^n (\mathrm{Int}(U_{\alpha_i}) \cup C) \subset \bigcup_{i=1}^n \mathrm{Int}_c(C \cap U_{\alpha_i}) = \bigcup_{i=1}^n \mathrm{Int}_c(V_{\alpha_i})$, and consequently C is RS-compact. Conversely, let $\{U_\alpha \mid \alpha \in I\}$ be a regular semiopen cover of X. Then $U_\alpha \cup (X - U_\alpha) = \mathrm{Int}(U_\alpha) \cup \mathrm{Cl}(X - U_\alpha) = X$ for each $\alpha \in I$. Let $\mathrm{Cl}(X - U_\alpha) = C$. Then $\{C \cap U_{\alpha_i} \mid \alpha \in I\}$ is a regular semiopen cover of C. By hypothesis, there exists a finite family such that $\bigcup_{i=1}^n \mathrm{Int}_c(C \cap U_{\alpha_i}) = C$. Therefore, $X = \mathrm{Int}(U_\alpha) \cup \mathrm{Cl}(X - U_\alpha) = \mathrm{Int}(U_\alpha) \cup \bigcup_{i=1}^n \mathrm{Int}_c(C \cap U_{\alpha_i}) \subset \mathrm{Int}(U_\alpha) \cup \bigcup_{i=1}^n (C \cap \mathrm{Int}(U_{\alpha_i})) = \mathrm{Int}(U_\alpha) \cup \bigcup_{i=1}^n \mathrm{Int}(U_{\alpha_i})$. Hence X is RS-compact.

We next give a general product theorem for RS-compact spaces.

THEOREM 2. 4. *Let $\{X_\alpha \mid \alpha \in I\}$ be a family of topological spaces. Then $\prod_\alpha X_\alpha$ is RS-compact if and only if X_α is RS-compact for each $\alpha \in I$.*

Proof. First, let $p_\beta : \prod_\alpha X_\alpha \to X_\beta$ be a projection and let $\{V_\beta{}^i \mid i \in I\}$ be a regular semiopen cover of X_β. Then $\{\prod_{\alpha \neq \beta} X_\alpha \times V_\beta{}^i \mid i \in I\}$ is a regular semiopen cover of $\prod_\alpha X_\alpha$. Since $\prod_\alpha X_\alpha$ is RS-compact, there exists a finite subfamily such that $\prod_\alpha X_\alpha = \bigcup_{j=1}^n \mathrm{Int}\ (\prod_{\alpha \neq \beta} X_\alpha \times V_\beta{}^i j) = \bigcup_{j=1}^n (\prod_{\alpha \neq \beta} X_\alpha \times \mathrm{Int}(V_\beta{}^i j))$.

By the projection p_β, $X_\beta = \bigcup_{j=1}^n \mathrm{Int}(V_\beta{}^i j)$. Therefore, X_β is RS-compact for each $\beta \in I$. Conversely, let $\{U_j = \prod_{\alpha \neq \alpha i_j} X_\alpha \times V_{\alpha 1_j} \times \cdots \times V_{\alpha n_j} \mid V_{\alpha i_j}$ is regular semiopen in $X_{\alpha i_j}$ for each $i \in \{1, 2, \dots, n\}$, $j \in I\}$ be a regular semiopen cover of $\prod_\alpha X_\alpha$. Then $\{p_\beta(U_j) \mid j \in I\}$ is a regular semiopen cover of X_β.

By hypothesis, there exists a finite subfamily such that $X_\beta = \bigcup_{k=1}^{m}\mathrm{Int}(p_\beta U_{jk})) = \bigcup_{k=1}^{m} p_\beta(\mathrm{Int}(U_{jk}))$. Case I. If $\prod_\alpha X_\alpha = \bigcup_{k=1}^{m}\mathrm{Int}(U_{jk})$, this completes the proof. Case II. If $\prod_\alpha X_\alpha \neq \bigcup_{k=1}^{m}\mathrm{Int}(U_{jk}))$, there exist at most finite $l_1, l_2, \ldots, l_h$ such that $X_{l_s} = \bigcup_{g=1^s}^{m}\mathrm{Int}\,(p_{l_s}(U_{jg}))$, for each $l_s \in \{l_1, l_2, \ldots, l_h\}$. Therefore, $\prod_\alpha X_\alpha = \bigcup_{k=1}^{m}\mathrm{Int}(U_{jk})) \cup \bigcup_{g=1}^{m_1}\mathrm{Int}\,(U_{jg}) \cup \ldots \cup \bigcup_{g=1}^{m_s}\mathrm{Int}(U_{jg})$, and hence $\prod_\alpha X_\alpha$ is *RS*-compact.

We can easily see that the *RS*-compactness is not, in general, preserved by continuous mappings. Our next theorem gives conditions when the *RS*-compactness is preserved under almost-continuous open mappings.

DEFINITION 2.5 [6]. A mapping $f: X \longrightarrow Y$ is called almost-continuous if the inverse images of regular open sets are open.

LEMMA 2.6 [5]. *Let $f: X \longrightarrow Y$ be an almost-continuous open mapping. Then for each open $A \subset Y$, $\mathrm{Cl}(f^{-1}(A)) \subset f^{-1}(\mathrm{Cl}(V))$.*

THEOREM 2.7. *If f is an almost-continuous open mapping of an RS-compact space X onto a topological space Y, then Y is RS-compact.*

Proof. Let $\{V_\alpha \mid \alpha \in I\}$ be a regular semiopen cover of Y. Then there is regular open A_α such that $A_\alpha \subset V_\alpha \subset \mathrm{Cl}(A_\alpha)$ for each $\alpha \in I$. Since f is almost-continuous open, $f^{-1}(A_\alpha) \subset f^{-1}(V_\alpha) \subset f^{-1}(\mathrm{Cl}(A_\alpha)) \subset \mathrm{Cl}(f^{-1}(A_\alpha))$. That is, $\{f^{-1}(V_\alpha) \mid \alpha \in I\}$ is a semiopen cover of X so $\{f^{-1}(V_\alpha) \cup \mathrm{Int}(\mathrm{Cl}(f^{-1}(V_\alpha)) \mid \alpha \in I\}$ is a regular semiopen cover of X. By hypothesis, there exists a finite subcover such that $X = \bigcup_{i=1}^{n}\mathrm{Int}\,(f^{-1}(V_{\alpha_i}) \cup \mathrm{Int}(\mathrm{Cl}(f^{-1}(V_{\alpha_i}))) = \bigcup_{i=1}^{n}\mathrm{Int}(\mathrm{Cl}(f^{-1}(V_{\alpha_i}))) = \bigcup_{i=1}^{n}\mathrm{Int}(\mathrm{Cl}(f^{-1}(A_{\alpha_i})))$. Therefore, it follows that $Y = \bigcup_{i=1}^{n} f(\mathrm{Int}(\mathrm{Cl}(f^{-1}(A_{\alpha_i}))))$. By Lemma 2.6 and hypothesis for f, $Y = \bigcup_{i=1}^{n} f(\mathrm{Int}(\mathrm{Cl}(f^{-1}(A_{\alpha_i})))) \subset \bigcup_{i=1}^{n}\mathrm{Int}(\mathrm{Cl}(f(f^{-1}(A_{\alpha_i})))) = \bigcup_{i=1}^{n}\mathrm{Int}(\mathrm{Cl}(A_{\alpha_i})) = \bigcup_{i=1}^{n} A_{\alpha_i} = \bigcup_{i=1}^{n}\mathrm{Int}(V_{\alpha_i})$. Hence Y is *RS*-compact.

COROLLARY 2.8. *If f is a continuous open mapping of an RS-compact space X onto a topological space Y, then Y is RS-compact.*

THEOREM 2.9. *If f is an almost-continuous open mapping of a topological space X onto an RS-compact space Y with $f^{-1}(f(A_\alpha)) \subset \mathrm{Cl}(A_\alpha)$ for each regular open sets A_α of X, then X is RS-compact.*

Proof. Let $\{V_\alpha \mid \alpha \in I\}$ be a regular semiopen cover of X. Then $\{f(V_\alpha) \mid \alpha \in I\}$ is a semiopen cover of Y so $\{f(V_\alpha) \cup \mathrm{Int}(\mathrm{Cl}(f(V_\alpha)) \mid \alpha \in I\}$ is a regular semiopen cover of Y. Since Y is *RS*-compact, there exists a finite subcover such that $Y = \bigcup_{i=1}^{n}\mathrm{Int}\,(f(V_{\alpha i}) \cup \mathrm{Int}(\mathrm{Cl}(f(V_{\alpha_i}))) = \bigcup_{i=1}^{n}\mathrm{Int}\,(\mathrm{Cl}(f(V_{\alpha_i}))) = \bigcup_{i=1}^{n}\mathrm{Int}(\mathrm{Cl}(f(A_{\alpha_i})))$. It follows that $X = \bigcup_{i=1}^{n}\mathrm{Int}(f^{-1}(\mathrm{Cl}(f(A_{\alpha_i})))$. By Lemma 2.6 and hypothesis for f, $X = \bigcup_{i=1}^{n}\mathrm{Int}(f(A_{\alpha_i}))) \subset$

$\bigcup_{i=1}^{n} \mathrm{Int}\,(\mathrm{Cl}(f^{-1}(f(A_{\alpha i})))\subset \bigcup_{i=1}^{n} \mathrm{Int}\,(\mathrm{Cl}(A_{\alpha i}))=\bigcup_{i=1}^{n} A_{\alpha_i}=\bigcup_{i=1}^{n} \mathrm{Int}\,(V_{i\alpha})$. Hence X is RS-compact.

Combining the results of Theorem2. 8 and 2. 9, we have the following corollaries whose proofs are omitted.

COROLLARY 2. 10. *Let $f: X \longrightarrow Y$ be an almost-continuous open bijection mapping. Then X is RS-compact if and only if Y is RS-compact.*

COROLLARY 2. 11. *Let $f: X \longrightarrow Y$ be an almost-continuous open mapping.*

(a) *If f is surjective and X is RS-compact, then Y is Quasi H-closed, nearly-compact and S-closed.*

(b) *If f is bijective and Y is RS-compact, then X is Quasi H-closed, nearly-compact and S-closed.*

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