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INFINITESIMAL VARIATIONS OF GENERIC SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD

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§ 0. Introduction

As is well known, a unit sphere S^{2m+1} of dimension $2m+1$ is a principal circle bundle over a complex projective space CP^m and Riemannian structure on CP^m is given by the submersion $\tilde{\pi}: S^{2m+1} \to CP^m$. This notation gives that fundamental properties of a submanifold would be applied to the study of real submanifolds of a complex projective space. Lawson [2], Maeda [3], Okumura [4] etc. have studied necessary or necessary and sufficient conditions for real hypersurfaces to be one of the model spaces $M_p{}^{,c}{}_q(a, b) = \tilde{\pi}(S^{2p+1}(a) \times S^{2q+1}(b))$, where (p, q) is some portion of $m-1$ and $a^2+b^2=1$.

On the other hand, Okumura [5] introduced the notion of generic submanifolds (anti-holomorphic submanifolds) in studying real submanifolds of codimension>1 in CP^m using the Hopf-fibration. In this paper, we consider a generic submanifold of codimension p of a Kaehlerian manifold and study infinitesimal variations which carry a generic submanifold into a generic submanifold. Such an infinitesimal variation will be called a generic variation.

The purpose of the present paper is to characterize a generic submanifold M in CP^m by taking account of the theory of Riemannian submersion when the generic variation preserves structure tensors. In determining a generic submanifold M in CP^m, we shall use the following theorems;

THEOREM A (Okumura [4]). *$M_p{}^{,c}{}_q(a, b)$ is the only hypersurface of a complex projective space in which the second fundamental tensor H commutes with the fundamental tensor F of the submersion.*

THEOREM B (Pak [6]) *Let M be a complete n-dimensional anti-holomorphic minimal submanifold of a complex projective space CP^m whose normal connection is flat. If the second fundamental tensor $h_{cb}{}^x$ of M satisfies*

$$h_{ce}{}^x f_b{}^e + h_{ce}{}^x f_b{}^e = 0,$$

then M is

$$\tilde{\pi}(S^{m_1}(r_1) \times \cdots \times S^{m_k}(r_k)),$$

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where $m_1, \dots, m_k$ *are odd numbers such that* $m_1, \dots, m_k \geqq 1$, $r_t = \sqrt{m_t/n+1}$ $(t=1, \dots, k)$, $n+1=\sum_{i=1}^{k} m_i$, $2m-n=k-1$.

Manifolds, submanifolds, geometric objects and mappings which are discussed in this paper are assumed to be differentiable and of C^∞. We use in the present paper the system of indices as follows;

$$h, i, j, k=1, 2, \dots, 2m; \quad a, b, c, d, e=1, 2, \dots, n;$$
$$w, x, y, z=1, 2, \dots, p; \qquad n+p=2m.$$

§ 1. Generic submanifolds of a Kaehlerian manifold

Let $\tilde{M}$ be a real $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, $F_i{}^h$ the almost comst complex structure tensor and g_{ji} the Hermitian metric tensor.

Then we have

$$(1.1) \qquad F_i{}^t F_t{}^h = -\delta_i^h, \qquad F_j{}^t F_i{}^s g_{ts} = g_{ji},$$

$$(1.2) \qquad \nabla_j F_i{}^h = 0,$$

where ∇_j denotes the operator of covariant differentiation with sespect to the Christoffel symbolr Γ_{ji}^h formed with g_{ji}

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and with metric tensor g_{cb}. We assume that M is isometrically immersed in $\tilde{M}$ by the immersion $i: M \rightarrow \tilde{M}$ and we identify $i(M)$ with M itself. We represent the immersion $i: M \rightarrow \tilde{M}$ locally by $x^h = x^h(y^a)$ and put $B_b{}^h = \partial_b x^h (\partial_b = \partial/\partial y^b)$, which are n linearly independent vectors of $\tilde{M}$ tangent to M. Then we have

$$(1.3) \qquad g_{cb} = g_{ji} B_c{}^j B_b{}^i$$

since the immersion is isometric.

We denote by $C_y{}^h$ $2m-n$ mutually orthogonal unit normals to M. Then the equations of Gauss are given by

$$(1.4) \qquad \nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h,$$

where ∇_c denotes the operator of van der Waerden-Bortolotti covariant differentiation along M and $h_{cb}{}^x$ are the second fundamental tensors of M with respect to the normal vectors $C_x{}^h$, and those of Weingarten by

$$(1.5) \qquad \nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h,$$

where $h_c{}^a{}_y = h_{cby} g^{ba} = h_{cb}{}^z g^{ba} g_{yz}$, g^{ba} being contravariant components of the metric tensor g_{cb} of M and g_{yx} the metric tensor of the normal bundle of M

defined by $g_{yx}=g_{ji}C_y{}^jC_x{}^i$. Thus equations of Gauss, Codazzi and Ricci are respectively given by

$$K_{dcb}{}^a=K_{kji}{}^hB_d{}^kB_c{}^jB_b{}^iB^a{}_h+h_d{}^a{}_xh_{cb}{}^x-h_c{}^a{}_xh_{db}{}^x, \tag{1.6}$$

$$K_{kji}{}^hB_d{}^kB_c{}^jB_b{}^iC^x{}_h=\nabla_dh_{cb}{}^x-\nabla_ch_{db}{}^x, \tag{1.7}$$

$$K_{dcy}{}^x=K_{kji}{}^hB_d{}^kB_c{}^jC_y{}^iC^x{}_h+h_{de}{}^xh_c{}^e{}_y-h_{ce}{}^xh_d{}^e{}_y, \tag{1.8}$$

where $B^a{}_h=B_b{}^jg^{ba}g_{jh}$, $C^x{}_h=C_y{}^jg^{yx}g_{jh}$, $K_{kji}{}^h$ is the curvature tensor of the ambient manifold $\tilde{M}$, $K_{dcb}{}^a$ and $K_{dcy}{}^x$ are those of the submanifold M and the normal bundle of M respectively.

If the transform by F of any normal vector to M is always tangent to M, that is, if there exists a tensor field $f_y{}^a$ of mixed type such that

$$F_i{}^hC_y{}^i=f_y{}^aB_a{}^h, \tag{1.9}$$

we say that M is *generic* (*anti-holomorphic*) in $\tilde{M}$ (cf. [5], [6]).

For the transform by F of tangent vectors $B_b{}^h$, we have equations of the form

$$F_i{}^hB_b{}^i=f_b{}^aB_a{}^h-f_b{}^xC_x{}^h, \tag{1.10}$$

where $f_b{}^a$ is a tensor field of type (1, 1) defined on M and we have put $f_b{}^x=f_y{}^ag_{ba}g^{yx}$.

Putting $f_{ba}=f_b{}^cg_{ca}$, $f_{ya}=f_y{}^bg_{ba}$ and $f_{ay}=f_a{}^xg_{xy}$, we can easily find

$$f_{ba}=-f_{ab},\quad f_{ay}=f_{ya}. \tag{1.11}$$

Applying F to (1.9) and (1.10) respectively and using (1.1) and these equations, we can easily verify

$$f_b{}^ef_e{}^a=-\delta_b^a+f_b{}^xf_x{}^a, \tag{1.12}$$

$$f_a{}^ef_e{}^x=0,\quad f_e{}^xf_b{}^e=0, \tag{1.13}$$

$$f_a{}^xf_y{}^a=\delta_y^x. \tag{1.14}$$

(1.12) and (1.13) show that M admits the so-called f-structure satisfying $f^3+f=0$.

Differentiating (1.9) and (1.10) covariantly along M respectively and using (1.2), (1.4) (1.5), and these equations, we find

$$\nabla_cf_b{}^a=h_{cb}{}^xf_x{}^a-h_{cx}{}^af_b{}^x, \tag{1.15}$$

$$\nabla_cf_b{}^x=h_{ce}{}^xf_b{}^e, \tag{1.16}$$

$$f_x{}^eh_{ce}{}^y=h_c{}^e{}_xf_e{}^y. \tag{1.17}$$

We now assume that the ambient manifold $\tilde{M}$ is of constant holomorphic sectional curvature c. Then it is well known that its curvature tensor $K_{kji}{}^h$ has the form

$$K_{kji}{}^h=\frac{c}{4}(\delta_k^h g_{ji}-\delta_j^h g_{ki}+F_k{}^h F_{ji}-F_j{}^h F_{ki}-2F_{kj}F_i{}^h). \tag{1.18}$$

Therefore, substituting (1.18) into (1.6), (1.7) and (1.8), we obtain that the equations of Gauss, Codazzi and Ricci are respectively given by

$$K_{dcb}{}^a=\frac{c}{4}(\delta_d^a g_{cb}-\delta_c^a g_{db}+f_d{}^a f_{cb}-2f_{dc}f_b{}^a)+h_d{}^a{}_x h_{cb}{}^x-h_c{}^a{}_x h_{db}{}^x, \tag{1.19}$$

$$\nabla_d h_{cb}{}^x-\nabla_c h_{db}{}^x=\frac{c}{4}(-f_d{}^x f_{cb}+f_c{}^x f_{db}+2f_{dc}f_b{}^x), \tag{1.20}$$

$$K_{dcy}{}^x=\frac{c}{4}(f_d{}^x f_{cy}-f_c{}^x f_{dy})+h_{de}{}^x h_c{}^e{}_y-h_{ce}{}^x h_d{}^e{}_y. \tag{1.21}$$

§2. Infinitesimal variations of generic submanifolds in a Kaehlerian manifold

We consider an infinitesimal variation of a generic submanifold M of a Kaehlerian manifold $\tilde{M}$ given by

$$\bar{x}^h=x^h(y)+v^h(y)\varepsilon, \tag{2.1}$$

where $v^h(y)$ is a vector field of $\tilde{M}$ defined along M and ε is an infinitesimal. Then we have

$$\bar{B}_b{}^h=B_b{}^h+(\partial_b v^h)\varepsilon, \tag{2.2}$$

where $\bar{B}_b{}^h=\partial_b\bar{x}^h$ are linearly independent vectors tangent to the varied submanifold.

We displace $\bar{B}_b{}^h$ parallelly from the varied point $(\bar{x}^h)$ to the original point (x^h). We then obtain the vectors

$$\tilde{B}_b{}^h=\bar{B}_b{}^h+\Gamma_{ji}^h(x+v\varepsilon)v^j\bar{B}_b{}^i\varepsilon$$

at the point (x^h), or

$$\tilde{B}_b{}^h=B_b{}^h+(\nabla_b v^h)\varepsilon, \tag{2.3}$$

neglecting the terms of order higher than one with respect to ε, where

$$\nabla_b v^h=\partial_b v^h+\Gamma_{ji}^h B_b{}^j v^i. \tag{2.4}$$

In the sequel we always neglect terms of order higher than one with respect to ε. Thus putting

$$\delta B_b{}^h=\tilde{B}_b{}^h-B_b{}^h, \tag{2.5}$$

which and (2.3) imply

(2. 6) $$\delta B_b{}^h = (\nabla_b v^h)\varepsilon.$$

Putting

(2. 7) $$v^h = v^a B_a{}^h + v^x C_x{}^h,$$

we have

(2. 8) $$\nabla_b v^h = (\nabla_b v^a - h_b{}^a{}_x v^x) B_a{}^h + (\nabla_b v^x + h_{ba}{}^x v^a) C_x{}^h$$

because of (1. 4) and (1. 5).

Now we denote by $\bar{C}_y{}^h$ $2m-n$ mutually orthogonal unit normal vectors to the varied submanifold and $\tilde{C}_y{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement from the point $(\bar{x}^h)$ to (x^h). Then we have

(2. 9) $$\tilde{C}_y{}^h = \bar{C}_y{}^h + \Gamma_{ji}{}^h(x+v\varepsilon) v^j \bar{C}_y{}^i \varepsilon.$$

We put

(2. 10) $$\delta C_y{}^h = \tilde{C}_y{}^h - C_y{}^h$$

and assume that $\delta\ C_y{}^h$ is of the form

(2. 11) $$\delta C_y{}^h = \eta_y{}^h \varepsilon = (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon.$$

Then, from (2. 9), (2. 10) and (2. 11), we have

(2. 12) $$\bar{C}_y{}^h = C_y{}^h - \Gamma_{ji}{}^h v^j C_y{}^i \varepsilon + (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon.$$

Applying the operator δ to $B_b{}^j C_y{}^i g_{ji} = 0$ and using (2. 6), (2. 8), (2. 11) and $\delta g_{ji} = 0$, we find

$$(\nabla_b v_y + h_{bay} v_a) + \eta_{yb} = 0,$$

where $v_y = v^z g_{yz}$ and $\eta_{yb} = \eta_y{}^c g_{cb}$, or

(2. 13) $$\eta_y{}^a = -(\nabla^a v_y + h_b{}^a{}_y v^b),$$

∇^a being defined to be $\nabla^a = g^{ac}\nabla_c$. Applying also the operator δ to $C_y{}^j C_z{}^i g_{ji} = g_{yz}$, and using (2. 11) and $\delta g_{ji} = 0$, we find

(2. 14) $$\eta_{yx} + \eta_{xy} = 0,$$

where $\eta_{yx} = \eta_y{}^z g_{zx}$.

We now assume that the infinitesimal variation (2. 1) carries a generic submanifold into a generic submanifold, that is,

(2. 15) $F_i{}^h(x+v\varepsilon)\bar{C}_x{}^i$ are linear combinations of $\bar{B}_b{}^h$.

Then, using $\nabla_j F_i{}^h = 0$ and (1. 6), we see that

$$\begin{aligned} F_i{}^h(x+v\varepsilon)\bar{C}_y{}^i &= (F_i{}^h + v^j \partial_j F_i{}^h \varepsilon)\{C_y{}^i - \Gamma_{kt}{}^i v^k C_y{}^t \varepsilon + (\eta_y{}^a B_a{}^i + \eta_y{}^x C_x{}^i)\varepsilon\} \\ &= \{F_i{}^h - v^j(\Gamma_{jt}{}^h F_i{}^t - \Gamma_{ji}{}^t F_t{}^h)\varepsilon\}\{C_y{}^i - \Gamma_{ks}{}^i v^k C_y{}^s \varepsilon + (\eta_y{}^a B_a{}^i + \eta_y{}^x C_x{}^i)\varepsilon\} \end{aligned}$$

$$=F_i^hC_y^i+\{F_i^h(\eta_y^aB_a^i+\eta_y^xC_x^i)-f_y^a\Gamma_{jt}^hv^jB_a^t\}\varepsilon,$$

that is, by (2. 2) and (2. 8),

$$F_i^h(x+v\varepsilon)\bar{C}_y^i=f_y^a\bar{B}_a^h-\{f_y^e(\nabla_ev^a-h_e^a{}_xv^x)+(\eta_y^ef_e^a+\eta_y^xf_x^a)\}\bar{B}_a^h\varepsilon-\{f_y^a(\nabla_av^x+h_{ae}{}^xv^e)+\eta_y^af_a^x\}\bar{C}_x^h\varepsilon. \tag{2.16}$$

Thus (2. 15) is equivalent to

$$f_y^a(\nabla_av^x+h_{ae}{}^xv^e)+\eta_y^af_a^x=0, \tag{2.17}$$

or, by (1. 15) and (2. 13), eqivalent to

$$f_y^a\nabla_av^x=f_a^x\nabla^av_y. \tag{2.18}$$

An infinitesimal variation given by (2. 1) is called a *generic variation* if it carries a generic submanifold into a generic submanifold. Thus we have

THEOREM 2. 1. *In order for an infinitesimal variation to be generic, it is necessary and sufficient that the variation vector v^h satisfies*

$$f_y^a\nabla_av^x=f_a^x\nabla^av_y.$$

COROLLARY 2. 2. *If a vector field v^h defines a generic variation, then another vector field v'^h which has the same normal part as v^h has the same property.*

For an infinitesimal variation given by (2. 1), when $v^x=0$, that is, when the variation vector v^h is tangent to the submanifold, we say that the variation is *tangential* and when $v^a=0$, that is, when the variation vector v^h is normal to the submanifold, we say that the variation is *normal.*

Then we have

THEOREM 2. 3. *A tangential variation is generic.*

Suppose that a generic variation given by (2. 1) carries a submanifold $x^h=x^h(y)$ into another submanifold $\bar{x}^h=\bar{x}^h(y)$ and the tangent space of the original submanifold at (x^h) and that of the varied submanifold at the corr esponding point $(\bar{x}^h)$ are parallel. Then we say that the variation is *parallel.*

Since we have from (2. 5), (2. 6) and (2. 8),

$$\tilde{B}_b^h=\{\delta_b^a+(\nabla_bv^a-h_b^a{}_xv^x)\varepsilon\}B_a^h+(\nabla_bv^x+h_{ba}{}^xv^a)C_x^h\varepsilon, \tag{2.19}$$

Thus we have

PROPOSITION 2. 4. ([8]) *In order for an infinitesimal variation to be parallel, it is necessary and sufficient that*

$$\nabla_bv^x+h_{ba}{}^xv^a=0. \tag{2.20}$$

If (2.20) is satisfied, then so is (2.18). Thus we have

THEOREM 2.5. *A parallel variation is a generic variation.*

§ 3. Variations of structure tensors

Suppose that an infinitesimal variation $\bar{x}^h = x^h + v^h(y)\varepsilon$ carries a generic submanifold into a generic submanifold, that is, it is generic. Then, putting

$$F_i{}^h(x+v\varepsilon)\bar{C}_y{}^i = (f_y{}^a + \delta f_y{}^a)\bar{B}_a{}^h, \tag{3.1}$$

from which and (2.16), we find

$$\delta f_y{}^a = \{\eta_y{}^x f_x{}^a - f_y{}^e(\nabla_e v^a - h_e{}^a{}_x v^x) - f_e{}^a(\nabla^e v_y + h_b{}^e{}_y v^b)\}\varepsilon. \tag{3.2}$$

Thus we have

PROPOSITION 3.1. *If an infinitesimal variation is generic, then the variation of $f_y{}^a$ is given by* (3.2).

PROPOSITION 3.2. *A generic variation preserves $f_y{}^a$ if and only if*

$$f_y{}^e(\nabla_e v^a - h_e{}^a{}_x v^x) + f_e{}^a(\nabla^e v_y + h_b{}^e{}_y v^b) - f_x{}^a \eta_y{}^x = 0. \tag{3.3}$$

We apply the operator δ to (1.10), and use $\delta F_i{}^h = 0$, (2.6) and (2.11). Then we get

$$F_i{}^h \nabla_b v^i \varepsilon = (\delta f_b{}^a)B_a{}^h + f_b{}^a \nabla_a v^h \varepsilon - (\delta f_b{}^x)C_x{}^h - f_b{}^x(\eta_x{}^a B_a{}^h + \eta_x{}^y C_y{}^h)\varepsilon.$$

If we substitute (2.8) into this equation, then we have

$$\begin{aligned}
&\{f_e{}^a(\nabla_b v^e - h_b{}^e{}_x v^x) + (\nabla_b v^x + h_{be}{}^x v^e) f_x{}^a\} B_a{}^h \varepsilon - f_e{}^x(\nabla_b v^e - h_b{}^e{}_y v^y) C_x{}^h \varepsilon \\
&= (\delta f_b{}^a)B_a{}^h + \{f_b{}^e(\nabla_e v^a - h_e{}^a{}_x v^x) - f_b{}^x \eta_x{}^a\} B_a{}^h \varepsilon - (\delta f_b{}^x)C_x{}^h \\
&\qquad + \{f_b{}^e(\nabla_e v^x + h_{ea}{}^x v^a) - f_b{}^y \eta_y{}^x\} C_x{}^h \varepsilon.
\end{aligned}$$

Comparing the tangential part and normal part of this, we have

$$\delta f_b{}^x = \{f_b{}^a(\nabla_a v^x + h_{ae}{}^x v^e) + f_a{}^x(\nabla_b v^a - h_b{}^a{}_y v^y) - f_b{}^y \eta_y{}^x\}\varepsilon, \tag{3.4}$$

$$\begin{aligned}
\delta f_b{}^a = \{&f_e{}^a(\nabla_b v^e - h_b{}^e{}_x v^x) - f_b{}^e(\nabla_e v^a - h_e{}^a{}_x v^y) + f_x{}^a(\nabla_b v^x + h_{be}{}^x v^e) \\
&- f_b{}^x(\nabla^a v_x + h_e{}^a{}_x v^e)\}\varepsilon.
\end{aligned} \tag{3.5}$$

We denote by $\mathcal{L}$ the Lie derivative with respect to v^a. Then (3.5) can be written as follows

$$\delta f_b{}^a = \{\mathcal{L} f_b{}^a - (f_e{}^a h_b{}^e{}_x - h_e{}^a{}_x f_b{}^e)v^x + f_x{}^a \nabla_b v^x - f_b{}^x \nabla^a v_x)\}\varepsilon. \tag{3.6}$$

PROPOSITION 3.3. *If an infinitesimal variation is generic, then the variation of $f_b{}^a$ is given by* (3.5) *or* (3.6).

PROPOSITION 3. 4. *A generic variation preserves the f-structure $f_b{}^a$ if and only if*

$$(3.7)\qquad \mathcal{L}f_b{}^a-(f_e{}^a h_b{}^e{}_x-h_e{}^a{}_x f_b{}^e)v^x+f_x{}^a\nabla_b v^x-f_b{}^x\nabla^a v_x=0.$$

Applying the operator δ to (1. 3) and using (2. 6), (2. 8) and $\delta g_{ji}=0$, we find

$$(3.8)\qquad \delta g_{cb}=(\nabla_c v_b+\nabla_b v_c-2h_{cb}{}^x v_x)\varepsilon,$$

from which,

$$(3.9)\qquad \delta g^{cb}=-(\nabla^c v^b+\nabla^b v^c-2h^{cb}{}_x v^x)\varepsilon.$$

A variation of a submanifold for which $\delta g_{cb}=0$ is said to be *isometric* and that for which δg_{cb} is proportional to g_{cb} is said to be *conformal*. Thus we have

PROPOSITION 3. 5. ([8]) *In order for an infinitesimal variation to be isometric or conformal, it is necessary and sufficient that*

$$(3.10)\qquad \nabla_c v_b+\nabla_b v_c-2h_{cbx}v^x=0,$$

or

$$(3.11)\qquad \nabla_c v_b+\nabla_b v_c-2h_{cbx}v^x=2\lambda g_{cb},$$

respectively, λ being a certain function given by

$$(3.12)\qquad \lambda=\frac{1}{n}(\nabla_a v^a-h_a{}^a{}_x v^x).$$

Now we assume that a generic variation preserves $f_y{}^a$, that is, $\delta f_y{}^a=0$. Transvecting (3. 3) with $f_a{}^z$ and using (1. 11) and (1. 12), we find

$$(3.13)\qquad \eta_{yz}=(\nabla_b v_a-h_{bax}v^x)f_y{}^b f_z{}^a.$$

Substituting (3. 13) into (2. 14), we get

$$(\nabla_b v_a+\nabla_a v_b-2h_{bax}v^x)f_y{}^b f_z{}^a=0.$$

If the variation is conformal, then we have $\lambda g_{yz}=0$ with the help of (3. 11), that is, $\lambda=0$. Thus we have

THEOREM 3. 6. *If a generic conformal variation preserves $f_y{}^a$, then it is isometric.*

Now we denote by g the determinant formed with g_{cb}. Then the volume element dV of M is given by

$$(3.14)\qquad dV=\sqrt{g}\,dy^1\wedge\cdots\wedge dy^n.$$

Since we have from (3.9) and (3.10)

$$\delta\sqrt{g}=\sqrt{g}\,(\nabla_e v^e-h_e{}^e{}_x v^x)\,\varepsilon,$$

from which, we get

$$\delta dV=(\nabla_e v^e-h_e{}^e{}_x v^x)\,dV\varepsilon. \tag{3.15}$$

Thus we have

PROPOSITION 3.7. ([8]) *In order for a variation of a submanifold to be volume-preserving, it is necessary and sufficient that*

$$\nabla_e v^e-h_e{}^e{}_x v^x=0.$$

PROPOSITION 3.8. ([8]) *In order for a normal variation of a submanifold to be volume-preserving, it is necessar and sufficient that*

$$h_e{}^e{}_x v^x y=0. \tag{3.16}$$

§ 4. Some characterizations of a generic submanifold of a complex projective space

In this section we assume that the ambient manifold of the submanifold is a complex projective space.

Suppose that the generic variation is normal and preserves the structure tensors $f_b{}^a$ and $f_y{}^a$. Then we have from (3.7)

$$(f_{ae}h_b{}^e{}_x+f_{be}h_a{}^e{}_x)v^x-(f_b{}^x\nabla_a v_x-f_a{}^x\nabla_b v_x)=0, \tag{4.1}$$

from which, taking the symmetric part with respect to a and b,

$$(f_{ae}h_b{}^e{}_x+f_{be}h_a{}^e{}_x)v^x=0, \tag{4.2}$$

and consequently

$$f_b{}^x\nabla_a v_x=f_a{}^x\nabla_a v_x. \tag{4.3}$$

Transvecting (4.3) with $f_c{}^b$, we find

$$f_a{}^x f_c{}^e\nabla_e v_x=0 \tag{4.4}$$

with the help of (1.11).

If we transvect (4.4) with $f_y{}^a$ and use (1.14), then we have

$$f_c{}^e\nabla_e v_x=0. \tag{4.5}$$

Substituting (4.5) into (3.3), we obtain

$$f_y{}^e h_e{}^a{}_x v^x+f_x{}^a n_y{}^x=0, \tag{4.6}$$

from which, transvecting with $f_a{}^z$ and using (1.14) and (2.14), we have

$$(h_e{}^a{}_x f_y{}^e)v^x=0. \tag{4.7}$$

We now assume that the generic variation admits $2m-n$ linearly independent normal variations. Consequently, we can see that (4.2) and (4.7) reduce respectively to

$$h_{ae}{}^x f_a{}^e + h_{ae}{}^x f_b{}^e = 0, \tag{4.8}$$

$$h_{ae}{}^x f_y{}^e = 0. \tag{4.9}$$

We now prove the following theorem by taking account of Theorem A in §0.

THEOREM 4.1. *Let M be an n-dimensional complete generic submanifold of CP^m. If the connection in the normal bundle of M is flat, and if $2m-n$ linearly independent generic normal variations preserve the structure tensors, then M is a real hypersurface of CP^m of the form*

$$M=M_{p},{}^{C}{}_{q}(a, b),$$

where (p, q) is some portion of $m-1$ and $a^2+b^2=1$.

Proof. Transvecting (1.21) with $f_w{}^d f_e{}^w$ and using (4.9), we find

$$f_b{}^x f_{cy} - f_c{}^x f_{by} = 0$$

because of $K_{dcy}{}^x=0$ and $c=4$. Transvection $f_z{}^b$ yields

$$\delta_z{}^x f_{cy} - f_c{}^x g_{zy} = 0$$

with the help of (1.14). We can see from this equation that the codimension $p=1$, that is, M is a real hypersurface of CP^m. Therefore, combining (4.8) with Theorem A in §0, it follows that $M=M_{p},{}^{C}{}_{q}(a, b)$, (p, q) being some portion of $m-1$ and $a^2+b^2=1$. Thus this theorem is proved.

On the other hand, by making use of Theorem B in §0 and (3.16), we obtain

THEOREM 4.2. *Let M be a complete n-dimensional generic submanifold of a complex projective space CP^m with flat normal connection. If $2m-n$ linearly independent generic normal variations preserve the f-structure and volume element of M, then M is of the form*

$$\tilde{\pi}(S^{m_1}(r_1)\times\cdots\times S^{m_k}(r_k)),$$

where $m_1, \dots, m_k$ *are odd numbers* ≥ 1, $r_t=\sqrt{m_t/n+1}$ $(t=1, \dots, k)$, $m_1+\cdots+m_k=n+1$, $2m-n=k-1$.

COROLLARY 4. 3. *Let M be the same submanifold as that stated in Theorem* 4. 2. *If $2m-n$ linearly independent parallel normal variations preserve the f-structure and volume element of M, then we have the same conclusion of Theorem* 4. 2.

Proof. The variation is evidently generic by means of Theorem 2. 5. But, differentiating (2. 20) with $v^a=0$ covariantly along M and taking account of the Ricci dientity, it must be that the connection in the normal bundle of M is flat. Thus, all assumptions in Theorem 4. 2 are satisfied and consequently we have the same conclusion of Theorem 4. 2.

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