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INFINITESIMAL VARIATIONS OF INVARIANT HYPERSURFACE OF A P-SASAKIAN MANIFOLD

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0. Introduction

An infinitesimal variation of an invariant submanifold of a Sasakian manifold which carries it into an invariant submanifold is said to be invariant. An infinitesimal variation is said to be f-preserving when it is invariant and preserves the induced tensor field $f_j{}^i$ of type (1, 1) on the invariant submanifold of a Sasakian manifold ([5]). K. Yano, U-H. Ki and J. S. Pak ([5]) proved that an infinitesimal fibre-preserving invariant conformal variation of a compact orientable invariant submanifold of a Sasakian manifold is necessarily f-preserving.

The main purpose of the present paper is to study infinitesimal variations of invariant hypersurfaces of a P-Sasakian manifold and to prove theorems analogous to those proved in [5].

In preliminary §1 we state some properties of invariant hypersurfaces of a P-Sasakian manifold. In §2, we derive fundamental formulas in the theory of infinitesimal variations and study invariant variations of hypersurfaces of a P-Sasakian manifold. In §3, we shall define f-preserving variations of invariant hypersurfaces of a P-Sasakian manifold. In the last §4, we shall study invariant conformal variations and prove that an invariant conformal fibre-preserving variation of a compact orientable hypersurface of a P-Sasakian manifold is necessarily isometric and hence f-preserving (see Theorem 4.3).

Throughout this paper, we assume that manifolds are orientable and every geometric object is differentiable.

1. Invariant hypersurfaces of a P-Sasakian manifold

Let M^n be an n-dimensional P-Sasakian manifold covered by a system of coordinate neighbourhoods $\{U, x^\lambda\}$ and $(\phi_\mu{}^\lambda, \xi^\lambda, \eta_\lambda, g_{\mu\lambda})$ the set of the structure tensors of M^n, where here and in the sequel, the indices $\nu, \mu, \dots, \lambda$ run over the range $\{1, 2, \dots, n\}$. Then we have by definition

$$(1.1)\qquad \begin{cases} \phi_\mu{}^\tau \phi_\tau{}^\lambda = \delta_\mu{}^\lambda - \eta_\mu \xi^\lambda, & \eta_\tau \phi_\mu{}^\tau = 0, \\ \phi_\tau{}^\lambda \xi^\tau = 0, & \eta_\tau \xi^\tau = 1 \end{cases}$$

$$(1.2)\qquad g_{\varepsilon\tau} \phi_\mu{}^\varepsilon \phi_\lambda{}^\tau = g_{\mu\lambda} - \eta_\mu \eta_\lambda,$$

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(1.3) $$\nabla_\nu \nabla_\mu \xi^\lambda = (-g_{\nu\mu}+\eta_\nu\eta_\mu)\xi^\lambda + (-\delta_\nu^\lambda+\eta_\nu\xi^\lambda)\eta_\mu,$$

where $\phi_\mu^\lambda=\nabla_\mu\xi^\lambda$, $\eta_\mu=g_{\mu\lambda}\xi^\lambda$ and the operator ∇_λ is the covariant differentiation with respect to $g_{\mu\lambda}$ ([3]).

Let M^{n-1} be an $(n-1)$-dimensional Riemannian manifold covered by a system of coordinate neighbourhoods $\{V, y^i\}$ and isometrically immersed in M^n by the immersion $l : M^{n-1}\to M^n$, where here and in the sequel, the indices $k, j, \dots, h$ run over the range $\{1, 2, \dots, n-1\}$. We identify $l(M^{n-1})$ with M^{n-1} and represent the immersion by $x^\lambda=x^\lambda(y^i)$. If we put $B_i^\lambda=\partial_i x^\lambda$ $(\partial_i=\partial/y^i)$, then B_i^λ are $n-1$ linearly independent vectors of M^n tangent to M^{n-1}. Denoting by g_{ji} the Riemannian metric of M^{n-1} we have $g_{ji}=g_{\mu\lambda}B_j^\mu B_i^\lambda$ since the immersion is isometric. We denote C^λ a unit normal to M^{n-1}.

The van der Waerden–Bortolotti covariant derivatives of B_i^λ and C^λ are respectively given by

$$\nabla_j B_i^\lambda=\partial_j B_i^\lambda+\Gamma_\nu{}^\lambda{}_\mu B_j^\nu B_i^\mu-\Gamma_j{}^h{}_i B_h^\lambda,$$
$$\nabla_j C^\lambda=\partial_j C^\lambda+\Gamma_\nu{}^\lambda{}_\mu B_j^\nu C^\mu,$$

and the equation of Gauss and Weingarten are respectively

(1.4) $$\nabla_j B_i^\lambda=h_{ji}C^\lambda, \qquad \nabla_j C^\lambda=-h_j^i B_i^\lambda,$$

where $\Gamma_\nu{}^\lambda{}_\mu$ and $\Gamma_j{}^h{}_i$ are the Christoffel symbols formed with $g_{\mu\lambda}$ and g_{ji} respectively and h_{ji} denote the components of the second fundamental tensor of M^{n-1} and $h_j^i=h_{jk}g^{ki}$, g^{ki} being contravariant components of the metric tensor of M^{n-1}.

A hypersurface M^{n-1} is called an invariant hypersurface of a P-Sasakian manifold M^n if the tangent space at each point of M^{n-1} is invariant under the action of ϕ_μ^λ. Thus for an invariant hypersurface M^{n-1}, we have

(1.5) $$\phi_\mu^\lambda B_j^\mu=f_j^i B_i^\lambda, \qquad \phi_\mu^\lambda C^\mu=\theta C^\lambda,$$

f_j^i being a tensor field of type $(1, 1)$ and θ a scalar field of M^{n-1}.

On the other hand, we put

(1.6) $$\xi^\lambda=f^i B_i^\lambda+\mu C^\lambda,$$

where f^i and μ are a vector field and a scalar field of M^{n-1} respectively.

Now applying the operator ϕ_λ^ν to the first equation of (1.5) and using (1.1) and (1.6), we have

(1.7) $$f_j^l f_l^i=\delta_j^i-f_j f^i, \qquad \mu f_i=0,$$

where $f_j=f^k g_{kj}$. Applying the operator ϕ_λ^ν to the second equation of (1.5) and using (1.1) and (1.6), we get

(1.8) $$f_i f^i=1-\mu^2, \qquad \mu^2+\theta^2=1.$$

Transvecting (1.5) with η_λ and taking account of (1.5), we find

(1.9) $$f_j^i f_i=0, \qquad \mu\theta=0.$$

By virtue of the second equation of (1.7) and (1.8), we have

(1.10) $$\mu(1-\mu^2)=0.$$

In the sequel, we consider the case of $\mu=0$, that is the P-Sasakian structure vector ξ^λ is tangent to the hypersurface. Then (1.6) and (1.8) can be written as

(1.11) $$\xi^\lambda=f^iB_i^\lambda,$$

(1.12) $$f_if^i=1,$$

respectively.

Also transvecting (1.2) with $B_j^\mu B_i^\lambda$, we get

(1.13) $$g_{lk}f_j^lf_i^k=g_{ji}-f_jf_i.$$

Next, differentiating the first equation of (1.5) covariantly along M^{n-1}, we can find

(1.14) $$\nabla_kf_j^i=(-g_{kj}+f_kf_j)f^i+(-\delta_k^i+f_kf^i)f_j,$$

(1.15) $$f_j^lh_{li}=h_{ji}.$$

We have from (1.15)

(1.16) $$f_j^lh_{li}=f_i^lh_{lj},$$

that is, the tensor field $f_j^lh_{li}$ is symmetric with respect to the indices j and i. Differentiating the second equation of (1.5) covariantly along M^{n-1} and taking account of $\mu=0$, (1.5) and (1.15), we have

(1.17) $$\theta=1.$$

Also differentiating (1.12) covariantly along M^{n-1}, we get

(1.18) $$\nabla_jf^i=f_j^i,$$

(1.19) $$h_{ji}f^i=0.$$

Thus from (1.7), (1.9), (1.12), (1.13), (1.14) and (1.18), we have

PROPOSITION 1.1. *Let M^{n-1} be an invariant hypersurface of a P-Sasakian manifold M^n, then the tensor field $(f_j^i, f^i, f_i, g_{ji})$ is a P-Sasakian structure on M^{n-1}.*

It is known that on a P-Sasakian manifold M^{n-1} the following identity is valid ([1]):

(1.20) $$K_{jh}-K_{jmli}f^{ml}f_h^i=(n-3)g_{jh}-(2n-5)f_jf_h-ff_{jh},$$

where K_{kji}^h and K_{ji} denote the curvature tensor and the Ricci tensor with respect to g_{ji} respectively and $f=f_j^j=\mathrm{trace}(f_j^i)$

2. Invariant variations of invariant hypersurfaces

Let M^{n-1} be an $(n-1)$-dimensional invariant hypersurface of an n-dimensional P-Sasakian manifold M^n. We consider an infinitesimal variation of M^{n-1} in M^n given by

(2.1) $$\bar{x}^\lambda=x^\lambda+v^\lambda(y)\varepsilon,$$

where ε is an infinitesimal. Putting $\bar{B}_i^\lambda=\partial_i\bar{x}^\lambda$, we have

(2.2) $$\bar{B}_i^\lambda=B_i^\lambda+(\partial_iv^\lambda)\varepsilon,$$

which are $n-1$ linearly independent vectors tangent to the varied hypersurface at $(\bar{x}^\lambda)$. We displace $\bar{B}_i{}^\lambda$ back parallely from $(\bar{x}^\lambda)$ to (x^λ) and put them $\tilde{B}_i{}^\lambda$, then we have

$$\tilde{B}_i{}^\lambda=\bar{B}_i{}^\lambda+\Gamma_\nu{}^\lambda{}_\mu(x+v\varepsilon)v^\nu\bar{B}_i{}^\mu\varepsilon\,.$$

Thus putting $\delta B_i{}^\lambda=\tilde{B}_i{}^\lambda-B_i{}^\lambda$, we obtain

(2.3) $$\delta B_i{}^\lambda=(\nabla_i v^\lambda)\varepsilon\,,$$

neglecting terms of order higher than one with respect to ε, where

(2.4) $$\nabla_i v^\lambda=\partial_i v^\lambda+\Gamma_\nu{}^\lambda{}_\mu B_i{}^\nu v^\mu.$$

Hereafter, we always neglect terms of order higher than one with respect to ε.

On the other hand, if we put

(2.5) $$v^\lambda=v^i B_i{}^\lambda+\alpha C^\lambda,$$

v^i and α being a vector field and a scalar field of M^{n-1} respectively, (2.4) can be written as

(2.6) $$\nabla_j v^\lambda=(\nabla_j v^i-\alpha h_j{}^i)B_i{}^\lambda+(\nabla_j\alpha+h_{ji}v^i)C^\lambda.$$

Substituting (2.6) in (2.3), we have

(2.7) $$\delta B_j{}^\lambda=[(\nabla_j v^i-\alpha h_j{}^i)B_i{}^\lambda+(\nabla_j\alpha+h_{ji}v^i)C^\lambda].$$

We now assume that the infinitesimal variation (2.1) carries an invariant hypersurface into an invariant hypersurface and call such a variation an infinitesimal invariant variation. For an infinitesimal invariant variation, $\phi_\mu(x+v\varepsilon)\bar{B}_j{}^\lambda$ are linear combination of $\bar{B}_j{}^\lambda$ and vice versa.

Now we can show that

(2.8) $$\begin{aligned}\phi_\mu{}^\lambda(x+v\varepsilon)\bar{B}_j{}^\mu&=[\phi_\mu{}^\lambda+v^\nu(\partial_\nu\phi_\mu{}^\lambda)\varepsilon][B_j{}^\mu+(\partial_i v^\mu)\varepsilon]\\&=[f_j{}^i+\{f_l{}^i(\nabla_j v^l-\alpha h_j{}^l)-f_j{}^l(\nabla_l v^i-\alpha h_l{}^i)\\&\quad-(v_j f^i+f_j v^i-2v^l f_l f_j f^i)\}\varepsilon]\bar{B}_i{}^\lambda-\{f_j{}^l(\nabla_l\alpha+h_{li}v^i)+\alpha f_j\}\bar{C}^\lambda\varepsilon.\end{aligned}$$

where $\bar{C}^\lambda$ denotes a unit normal to the varied hypersurface and $v_i=v^h g_{hi}$. Thus using (1.15) and (2.8), we have

THEOREM 2.1. *In order for an infinitesimal variation of an invariant hypersurface of a P-Sasakian manifold to be invariant it is necessary and sufficient that*

(2.9) $$f_i{}^l(\nabla_l\alpha+h_{lk}v^k)+\alpha f_i=0$$

or

(2.10) $$f_i{}^l\nabla_l\alpha+h_{il}v^l=0.$$

Transvecting (2.9) with f^i and using (1.9) and (1.13), we have

(2.11) $$\alpha=0.$$

Thus we have

THEOREM 2.2. *If an infinitesimal variation of an invariant hypersurface of a P-Sasakian manifold is invariant, then the infinitesimal variation is tangential.*

When the tangent space at a point (x^λ) of a hypersurface and that at the corresponding point $(\bar{x}^\lambda)$ of the varied hypersurface are always parellel, the variation is said to be parallel ([5]). The following lemma was proved by K. Yano:

LEMMA 2.3 ([4]). *In order for an infinitesimal variation* (2.1) *of a hypersurface to be parallel, it is necessary and sufficient that*

(2.12) $$\nabla_j \alpha + h_{ji} v^i = 0.$$

Thus we have from Theorem 2.1 and Lemma 2.3

COROLLARY 2.4. *In order for a infinitesimal parallel variation of an invariant hypersurface of a P-Sasakian manifold to be invariant, it is necessary and sufficient that the variation is tangential.*

Next, applying the operator δ to $g_{ji} = g_{\mu\lambda} B_j{}^\mu B_i{}^\lambda$ and taking account of $\delta g_{\mu\lambda} = 0$ and (2.7), we have

(2.13) $$\delta g_{ji} = (\nabla_j v_i + \nabla_i v_j - 2\alpha h_{ji})\varepsilon,$$

from which

(2.14) $$\delta g^{ji} = -(\nabla^j v^i + \nabla^i v^j - 2\alpha h^{ij})\varepsilon.$$

An infinitesimal variation for which $\delta g_{ji} = 0$ is said to be isometric ([4]).

We put

(2.15) $$\bar{\Gamma}_j{}^h{}_i = (\partial_j \bar{B}_i{}^\lambda + \Gamma_\nu{}^\lambda{}_\mu (x + v\varepsilon) \bar{B}_j{}^\nu \bar{B}_i{}^\mu) \bar{B}^h{}_\lambda$$

and

(2.16) $$\delta \Gamma_j{}^h{}_i = \bar{\Gamma}_j{}^h{}_i - \Gamma_j{}^h{}_i,$$

where $\bar{\Gamma}_j{}^h{}_i$ denote the Christoffel symbols of the varied hypersurface. Then we can find by straightfoward computation

(2.17) $$\delta \Gamma_j{}^h{}_i = [(\nabla_j \nabla_i v^\lambda + K_{\omega\nu\mu}{}^\lambda v^\omega B_j{}^\nu B_i{}^\mu) B^h{}_\lambda + h_{ji}(\nabla^h \alpha + h_l{}^h v^l)],$$

where $K_{\omega\nu\mu}{}^\lambda$ is the curvature tensor with respect to $g_{\mu\lambda}$. By virtue of the equations of Gauss and Codazzi, (2.5) and (2.6), (2.17) can be written as

(2.18) $$\delta \Gamma_j{}^h{}_i = [(\nabla_j \nabla_i v^h + K_{kji}{}^h v^k)\varepsilon - \{\nabla_j(\alpha h_{il}) + \nabla_i(\alpha h_{jl}) - \nabla_l(\alpha h_{ji})\} g^{lh}\varepsilon.$$

An infinitesimal variation for which $\delta \Gamma_j{}^h{}_i = 0$ is said to be affine.

Since an infinitesimal isometric variation is affine, for an infinitesimal isometric variation, we have from (2.18)

(2.19) $$\nabla_j \nabla_i v_h + K_{kjih} v^k - \{\nabla_j(\alpha h_{ih}) + \nabla_i(\alpha h_{jh}) - \nabla_h(\alpha h_{ji})\} = 0,$$

from which

(2.20) $$\nabla^j \nabla_j v_i + K_{ki} v^k - \{2\nabla_j(\alpha h_i{}^j) - \nabla_i(\alpha h)\} = 0,$$

where $h = \mathrm{trace}(h_j{}^i)$.

Substituting (1.20) in (2.20), we have

(2.21) $$\nabla^j \nabla_j v_i + \{K_{ktsm} f^{ts} f_i{}^m + (n-3) g_{ki} - (2n-5) f_k f_i - f f_{ki}\} v^k - \{2\nabla_j(\alpha h_i{}^j) - \nabla_i(\alpha h)\} = 0.$$

Thus we have

THEOREM 2.4. *If an infinitesimal variation* (2.1) *of an invariant hypersurface of a P–Sasakian manifold is isometric, then* (2.21) *is valid.*

3. Infinitesimal *f*-preserving variations.

Let an infinitesimal variation (2.1) be invariant and put

(3.1) $\phi_\mu{}^\lambda(x+v\varepsilon)\bar{B}_i{}^\mu=(f_i{}^h+\delta f_i{}^h)\bar{B}_h{}^\lambda.$

Then, by virtue of (2.8) and (3.1), we have

(3.2) $\delta f_i{}^h=\{f_l{}^h\nabla_i v^l-f_i{}^l\nabla_l v^h+2v^l f_l f_i f^h-v_i f^h-f_i v^h\}\varepsilon.$

An infinitesimal invariant variation for which $\delta f_j{}^i=0$ is said to be *f*–preserving.

From (2.10) and (3.2), we have

THEOREM 3.1. *In order for an infinitesimal variation* (2.1) *of an invariant hypersurface of a P–Sasakian manifold to be f–preserving, it is necessary and sufficient that the variation satisfies* (2.10) *and*

(3.3) $f_l{}^h\nabla_i v^l-f_i{}^l\nabla_l v^h+2v^l f_l f_i f^h-v_i f^h-f_i v^h=0$

or equivalently

(3.4) $\theta(v)f_i{}^h=0,$

where $\theta(v)$ *denotes the Lie derivative with respect to* v^i.

Now, applying the operator δ to $f_j{}^i f^j=0$ and using (3.2), we can find

(3.5) $\delta f^h=(\theta(v)f^h+\beta f^h)\varepsilon$

for a certain scalar field β on M^{n-1}. On the other hand, applying the operator δ to $g_{ji}f^j f^i=1-\mu^2$ and the second equation of (1.7), we respectively have

$$(\delta g_{ji})f^j f^i+2g_{ji}(\delta f^j)f^i=-2\mu\delta\mu$$

and

$$(\delta\mu)f_i+\mu\delta f_i=0.$$

From the above two equations, (1.8), (1.20) and (2.13), if the *P*–Sasakian structure vector ξ^λ is tangent to M^{n-1}, that is, $\mu=0$, we have

$$(\theta(v)g_{ji})f^j f^i+2g_{ji}(\delta f^j)f^i=0,\qquad \delta\mu=0.$$

Substituting (3.5) in the first equation of the above equation, we get $\beta=0$. Thus we have

(3.6) $\delta f^h=(\theta(v)f^h)\varepsilon.$

Next, we define a tensor field T_{ji} by

(3.7) $T_{ji}=\nabla_j v_i-f_j{}^l f_i{}^k\nabla_l v_k-v_l f_i{}^l f_j+v_l f_j{}^l f_i.$

Then we now prove

THEOREM 3.2. *In order for an infinitesimal isometric invariant variation* (2.1) *of an invariant hypersurface of a P–Sasakian manifold to be f–preserving, it is necessary and sufficient that* $T_{ji}=0$.

Proof. Suppose that an infinitesimal variation of an invariant hypersurface is f-preserving. Then by Theorem 3.1, we have (3.3). Transvecting (3.3) with $f_k{}^j$, we get

(3.8) $\nabla_j v_i - f_i f^l \nabla_j v_l - f_j{}^l f_i{}^k \nabla_l v_k - f_j v_l f_i{}^l = 0.$

Substituting (3.8) into (3.7), we have

(3.9) $T_{ji} = v_l f_j{}^l f_i + f_i f^l \nabla_j v_l.$

Transvecting (3.8) with f^j, we get $f^l \nabla_i v_l = -v_l f_i{}^l$.

Thus we have from the above equation and (3.9) $T_{ji}=0$.

Conversely, we assume that $T_{ji}=0$. Then we have

$$\nabla_j v^i - f_k{}^i f_j{}^l \nabla_l v^k - v^l f_l{}^i f_j + v_l f_j{}^l f^i = 0.$$

Transvecting the above equation with $f_i{}^h$ and f^j, we respectively have

(3.10) $f_l{}^h \nabla_j v^l - f_j{}^l \nabla_l v^h + f_j{}^l f_k f^h \nabla_l v^k - v^h f_j + v^l f_l f_j f^h = 0,$

(3.11) $f^l \nabla_l v_i = v^l f_{li}.$

On the other hand, since the infinitesimal variation is isometric, we have from (3.11)

(3.12) $f^l \nabla_i v_l = -v^l f_{li}.$

Substituting (3.12) in (3.10), we find $\delta f_j{}^i = 0$.

Next, we shall prove

LEMMA 3.3. *For an infinitesimal isometric invariant variation* (2.1) *of an invariant hypersurface of a P-Sasakian manifold, we have*

(3.13) $T_{ji} + T_{ij} = 0,$

(3.14) $T_{ji} + f_j{}^l f_i{}^k T_{lk} = f_j \theta(v) f_i - f_i \theta(v) f_j,$

(3.15) $T_{ji} T^{ji} = 2T^{ji} \nabla_j v_i + 2(\theta(v) f_i)(\theta(v) f^i).$

Proof. By the definition of T_{ji}, (3.13) is clear.

Next, we have

$$T_{lk} f_j{}^l f_i{}^k = -\nabla_j v_i + f_j{}^l f_i{}^k \nabla_l v_k + f_j v^l f_{li} - f_i v^l f_{lj} + f_i \theta(v) f_j - f_j \theta(v) f_i,$$

where we used the identity $f^l \nabla_j v_l = \theta(v) f_j - v^l f_{lj}$.

Thus, using (3.7) and the above equation, we have (3.14).

Finally, from (3.7), (3.13) and (3.14), we have

$$T^{ji} T_{ji} = 2T^{ji} \nabla_j v_i - (f_l \theta(v) f_k - f_k \theta(v) f_l) \nabla^l v^k - v_l f_i{}^l (T^{ji} f_j) + v_l f_j{}^l (T^{ji} f_i).$$

Transvecting (3.7) with f_j, we get

$$T^{ji} f_j = -\theta(v) f^i.$$

So we obtain

$$T^{ji} T_{ji} = 2T^{ji} \nabla_j v_i + 2 g_{il} f_k (\theta(v) f^i) \nabla^l v^k + 2(\theta(v) f^i) v_l f_i{}^l.$$

Substituting $g_{il} f_k \nabla^l v^k = \theta(v) f_i - v^l f_{li}$ into the above equation, we get (3.15).

Next, applying the operator ∇^j to (3.7) and taking account of (1.14), we have

$$\nabla^j T_{ji} = \nabla^j \nabla_j v_i + f_i{}^l f^k \nabla_l v_k + (n-3) f_i{}^k f^l \nabla_l v_k$$

$$-f_i{}^k f^{jl}\nabla_j\nabla_l v_k - f v_l f_i{}^l + v_i - (n-1)v^l f_l f_i.$$

Since an infinitesimal isometric variation is affine, substituting (2.19) with $\alpha=0$ into the above equation and using (1.20) and (2.20) with $\alpha=0$, we obtain

$$(3.16)\qquad (\nabla^j T_{ji})v^i = (n-4)\{f_i{}^k f^l v^i \nabla_l v_k - v_i v^i + (v_i f^i)^2\}.$$

Thus we have from (3.15) and the above equation

$$(3.17)\qquad \nabla^j (T_{ji}v^i) = (n-4)\{f_i{}^k f^l v^i \nabla_l v_k - v_i v^i + (v_i f^i)^2\} + \frac{1}{2}T^{ji}T_{ji} - (\theta(v) f_i)(\theta(v) f^i).$$

Now, an infinitesimal variation which satisfies $\theta(v)f^i=\tau f^i$, τ being a certain scalar field on M^{n-1} is said to be fibre-preserving. Thus, if an infinitesimal isometric invariant variation is fibre-preserving, we get $\theta(v)g_{ji}=0$ and $\theta(v)f^i=\tau f^i$.

Furthermore we get

$$\theta(v)f_i=\theta(v)(f^j g_{ji}) = (\theta(v)f^j)g_{ji} + f^j\theta(v)g_{ji} = \tau f_i.$$

Apllying the operator $\theta(v)$ to $f_i f^i=1$, we get $\tau=0$.

Hence we have

THEOREM 3.4. *For an infinitesimal isometric invarient variation of an invariant hypersurface of a P-Sasakian manifold, if the variation is fibre-preserving, then we have*

$$(3.18)\qquad \theta(v)f^i=0,$$

that is, the vector field v^i is a strict paracontact vector field defined in [2].

Next, since

$$\theta(v)f_j{}^i = -v_j f^i - v^i f_j + 2 f_l v^l f_j f^i - f_j{}^l \nabla_l v^i + f_l{}^i \nabla_j v^l$$

we get

$$(3.19)\qquad (\theta(v)f_j{}^i)f^j = -v^i + f_l v^l f^i + f_l{}^i f^k \nabla_k v^l.$$

Thus, for an infinitesimal isometric invariant fibre-preserving variation, we have from (3.18) and (3.19)

$$\nabla^j (T_{ji}v^i) = \frac{1}{2}T^{ji}T_{ji},$$

from which, if the hypersurface is compact orientable, we have

$$\int_{M^{n-1}} (T^{ji}T_{ji})\,dV = 0,$$

dV being the volume element of M^{n-1}. Thus we have

THEOREM 3.5. *If an infinitesimal isometric invariant variation of a compact orientable invariant hypersurface of a P-Sasakian manifold is fibre-preserving, then it is f-preserving.*

4. Infinitesimal conformal variations

An infinitesimal variation of a hypersurface for which δg_{ji} is proportional to g_{ji} is said to be conformal. A necessary and sufficient condition for an infinitesimal variation (2.1) of an invariant hypersurface of a P-Sasakian manifold to be conformal is

(4.1) $$\nabla_j v_i + \nabla_i v_j - 2\alpha h_{ji} = 2\lambda g_{ji},$$

where

(4.2) $$\lambda = \frac{1}{n-1}(\nabla_j v^j - \alpha h).$$

Now, we assume that an infinitesimal variation is a conformal invariant variation. Then we have from (2.11) and (4.1)

(4.3) $$\nabla_j v_i + \nabla_i v_j = 2\lambda g_{ji}.$$

We define a tensor field $\bar{T}_{ji}$ by

(4.4) $$\bar{T}_{ji} = \nabla_j v_i - f_j{}^l f_i{}^k (\nabla_l v_k) - f^l f^k (\nabla_l v_k) f_j f_i - f_j f_i{}^l v_l + f_i f_j{}^l v_l.$$

Next Theorem and Lemma will be proved in the same way as in proofs of Theorem 3.2 and Lemma 3.3. So we omit their proofs.

THEOREM 4.1. *In order for an infinitesimal conformal invariant variation* (2.1) *of an invariant hypersurface of a P–Sasakian manifold to be f–preserving, it is necessary and sufficient that the tensor field $\bar{T}_{ji}$ defined by* (4.4) *vanishes identically.*

LEMMA 4.2. *For an infinitesimal conformal invariant variation of an invariant hypersurface of a P–Sasakian manifold, we have*

(4.5) $$\bar{T}_{ji} + \bar{T}_{ij} = 0,$$

(4.6) $$\bar{T}_{ji} + \bar{T}_{lk} f_j{}^l f_i{}^k = f_i \theta(v) f_j - f_j \theta(v) f_i,$$

(4.7) $$\bar{T}^{ji}\bar{T}_{ji} = 2\bar{T}^{ji}\nabla_j v_i + 2(\theta(v) f_j)(\theta(v) f^j) + 4(\theta(v) f_l) f_k{}^l v^k - 2\lambda f^l \theta(v) f_l.$$

For an infinitesimal conformal variation, we have from (4.1)

(4.8) $$\nabla_j \nabla_i v_h - K_{ihjk} v^k = \lambda_j g_{ih} + \lambda_i g_{hj} + \lambda_h g_{ji},$$

from which

(4.9) $$\nabla^j \nabla_j v_h + K_{kh} v^k = -(n-3)\lambda_h,$$

where we put $\lambda_h = \partial_h \lambda$.

Now applying the operator ∇^j to (4.4), we find

$$\nabla^j \bar{T}_{ji} = \nabla^j \nabla_j v_i + K_{tjlk} v^t f^{jl} f_i{}^k - 2\lambda_j g_{lk} f^{jl} f_i{}^k + f\lambda_k f_i{}^k + (n-2) f^l (\nabla_l v_k) f_i{}^k + f^k (\nabla_l v_k) f_i{}^l + (\nabla_l v_k) f^{lk} f_i - \nabla^j (\lambda f_j f_i) - f f_i{}^l v_l - f^j f_i{}^l \nabla_j v_l + v_i - (n-1) f^l v_l f_i + \lambda f f_i.$$

Substituting (1.21) in the above equation and taking account of (4.1) and the identity

$$f^l (\nabla_k v_l) f_i{}^k = (\theta(v) f_k) f_i{}^k - v_i + f^l v_l f_i,$$

we have

$$\nabla^j \bar{T}_{ji} = \nabla^j \nabla_j v_i + K_{ti} v^t - 2\lambda_i + 2\lambda_l f^l f_i + f\lambda_k f_i{}^k - (n-4)(\theta(v) f_l) f_i{}^l + 2\lambda f f_i - \nabla^j (f_j f_i).$$

Furthermore, substituting (4.9) in the above equation, we get

$$\nabla^j\bar{T}_{ji}=-(n-1)\lambda_i+2\lambda_l f^l f_i+f\lambda_l f_i^l-(n-4)(\theta(v)f_l)f_i^l+2\lambda f f_i$$
$$-\nabla^j(\lambda f_j f_i).$$

Since

$$\nabla^j(\lambda f_j f_i)=\lambda_l f^l f_i+\lambda f f_i,$$

the above equation can be written as

$$\nabla^j\bar{T}_{ji}=-(n-1)\lambda_i+\nabla^j(\lambda f_j f_i)+f\lambda_l f_i^l-(n-4)(\theta(v)f_l)f_i^l.$$

Substituting $f\lambda_l f_i^l=f\nabla^j(\lambda f_{ji})+(n-2)\lambda f f_i$ into the above equation, we have

(4.10) $\nabla^j(\bar{T}_{ji}+(n-1)\lambda g_{ji}-\lambda f_j f_i+\lambda f f_{ji})=(n-2)\lambda f f_i-(n-4)(\theta(v)f_l)f_i^l.$

By virtue of (4.7) and (4.10), we obtain

(4.11) $$\nabla^j\{(\bar{T}_{ji}+(n-1)\lambda g_{ji}-\lambda f_j f_i-\lambda f f_{ji})v^i\}$$
$$=(n-2)\{\lambda f f_i-(\theta(v)f_l)f_i^l\}v^i+\frac{1}{2}\bar{T}^{ji}\bar{T}_{ji}-(\theta(v)f_l)(\theta(v)f^l)$$
$$+\lambda f^l\theta(v)f_l+(n-1)^2\lambda^2-\lambda^2(1-f^2).$$

Next, we assume that our variation is fibre-preserving. Then it follows from $\theta(v)g_{ji}=2\lambda g_{ji}$ that

$$\theta(v)f_i=f_i,\qquad \theta(v)f^i=-\lambda f^i.$$

Substituting the above equations in (4.11), we have

(4.13) $$\nabla^j\{(\bar{T}_{ji}+(n-1)\lambda g_{ji}-\lambda f_j f_i-\lambda f f_{ji})v^i\}$$
$$=(n-2)\lambda f f_i v^i+\frac{1}{2}\bar{T}^{ij}\bar{T}_{ji}+\{1-f^2+(n-1)^2\}\lambda^2.$$

On the other hand, by virtue of (4.2) with $\alpha=0$, we have

$$(n-2)\lambda f f_i v^i=\frac{n-2}{n-1}(\nabla_j v^j)f f_i v^i=\frac{n-2}{n-1}\{\nabla^j(f f_i v^i v_j)-f v^j\theta(v)f_j\}$$
$$=\frac{n-2}{n-1}\{\nabla^j(f f_i v^i v_j)-\lambda f v_j f^j\}.$$

Thus we have

(4.14) $$(n-2)\lambda f f_j v^j=\frac{n-2}{n}\nabla^j(f f_i v^i v_j).$$

Substituting (4.14) in (4.13), we obtain

(4.15) $$\nabla^j\{(\bar{T}_{ji}+(n-1)\lambda g_{ji}-\lambda f f_j f_i-\lambda f f_{ji}-\frac{n-2}{n}f v_j f_i)v^i\}$$
$$=\frac{1}{2}\bar{T}^{ji}\bar{T}_{ji}+\{1-f^2+(n-1)^2\}\lambda^2.$$

Thus if the hypersurface is compact orientable, we have

(4.16) $$\int_{M^{n-1}}[\frac{1}{2}\bar{T}^{ji}\bar{T}_{ji}+\{1-f^2+(n-1)^2\}\lambda^2]dV=0.$$

On the other hand, in a P-Sasakian hypersurface M^{n-1} we can easily show that $f^2<(n-1)^2$.

So we have from (4.16) $\bar{T}_{ji}=0$ and $\lambda=0$.

Thus we have

THEOREM 4. 3. *If an infinitesimal conformal invariant variation of a compact orientable hypersurface of a P-Sasakian manifold is fibre-preserving, then it is isometric and hence f-preserving.*

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