J. Korean Math. Soc.
Vol. 16, No. 2, 1980

A CERTAIN POLYNOMIAL STRUCTURE

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0. Introduction

K. Matsumoto[8] has introduced the pseudo-f-structure defined by a tensor field f of type $(1,1)$ satisfying $f^3-f=0$ and investigated the integrability conditions of the pseudo-f-structure. On the other hand, I. Sato [11] has studied an almost paracontact structure (f, ξ, η) of the pseudo-f-structure of rank n-1. The purpose of the present paper is to introduce a pseudo-framed structure and to obtain the results analogous to the properties of a framed structure. In § 1 we introduce a pseudo-framed structure of rank r and give an example of a manifold with such a structure. This structure is a generalization of an almost product structure and almost paracontact structure.

In § 2 we study structures induced on a product manifold of two pseudo-framed manifolds and prove the manifold $M\times R^{n-r}$ has an almost product structure. In § 3 we define the normal pseudo-framed structure and prove that the product manifold of two normal pseudo-framed manifolds has a normal pseudo-framed structure.

1. Pseudo-framed structure

Let M be an n-dimensional differentiable manifold of class C^∞. If there exists a tensor field f of type $(1,1)$ of constant rank r satisfying the polynomial equation:

(1.1) $$f^3-f=0,$$

then we call the structure a pseudo-f-structure of rank r and the manifold M pseudo-f-manifold of rank r ([8]). This structure is a generalization of an almost product structure $(r=n)$ and almost paracontact structure $(r=n-1)$ ([11]).

If we put

(1.2) $$s=f^2, \qquad t=-f^2+I,$$

where I is the identity transformation field, then we get

(1.3) $$s+t=I, \qquad s^2=s, \qquad t^2=t,$$
$$fs=f, \qquad ft=0, \qquad st=0.$$

Received Jan. 7, 1980
*This research was supported by the Korea Science and Engineering Foundation.

The operators s and t acting in the tangent space at each point of M are therefore complementary projection operators and there exist complementary distributions S and T corresponding to the operators s and t, respectively. Then the distribution S is r-dimensional and distribution T is $(n-r)$-dimensional.

Let M be a manifold with pseudo-f-structure of rank r. There exist $n-r$ vector fields ξ_x spanning the distribution T and its dual 1-forms η_x, where the indicies x, y, z run over the range $\{1, 2, \ldots, n-r\}$. Then we can put

(1.4) $$t=\eta_x \otimes \xi_x, \qquad \eta_x(\xi_y)=\delta_{xy}$$

where δ_{xy} is the Kronecker's delta, the summation convention being employed here and in the sequel. Therefore, for any vector field X we have

(1.5) $$sX=f^2X, \qquad tX=\eta_x(X)\xi_x,$$

from which

(1.6) $$f^2=I-\eta_x \otimes \xi_x.$$

From (1.3) and (1.5) we easily see that

(1.7) $$f\xi_x=0, \qquad \eta_x \circ f=0.$$

If there exist on M vector fields ξ_x and 1-forms η_x satisfying (1.4), (1.6) and (1.7), then the set (f, ξ_x, η_x) is called a pseudo-f-structure with complementary frame, or simply, a pseudo-framed structure and the manifold M a pseudo-framed manifold.

Let M be a manifold with pseudo-framed structure of rank r. Then there exists on M a Riemannian metric g such that

(1.8) $$g(X, \xi_x)=\eta_x(X),$$

(1.9) $$g(fX, fY)=g(X, Y)-\eta_x(X)\eta_x(Y).$$

for any vector fields X and Y on M.

If we put

(1.10) $$F(X, Y)=g(X, fY),$$

then we get

(1.11) $$F(X, Y)=F(Y, X),$$

which shows that F is a symmetric tensor.

Now, as an example, we consider a submanifold N of codimension r of an n-dimensional almost product manifold M with structure tensor (J, G). If B denotes the differential of imbedding $i: N \to M$ and X and Y are any vector fields of N, then the induced metric g on N is defined by

(1.12) $$g(X, Y)=G(BX, BY).$$

We assume that the normal bundle of N is orientable. Then we choose mutually orthogonal unit vector fields C_x normal to N.

The transformations JBX and JC_x can be expressed as

(1.13) $$JBX=BfX+\eta_x(X)C_x,$$

(1.14) $$JC_x=B\xi_x+\lambda_x C_x,$$

where f is a tensor field of type (1.1), η_x are 1-forms, ξ_x are vector fields and λ_x are scalar fields defined on N.

We are interested in the antinormal submanifold, that is, $\lambda_x=0$ in (1.14). Then computing J^2BX, we get

$$BX=Bf^2X+\eta_x(fX)C_c+\eta_x(X)B\xi_x,$$

from which, comparing tangential part and normal part,

$$f^2X=X-\eta_x(X)\xi_x, \qquad \eta_x(fX)=0.$$

Similarly, computing J^2C_x we get

$$f\xi_x=0, \qquad \eta_y(\xi_x)=\delta_{yx}.$$

Therefore the antinormal submanifold N has a pseudo-framed structure of rank r.

2. Products of pseudo-framed manifolds

Let $M(f, \xi_x, \eta_x)$ and $\overline{M}(\bar{f}, \bar{\xi}_\alpha, \bar{\eta}_\alpha)$ be two pseudo-framed manifolds of ranks r and $\bar{r}$, respectively, where the index x runs over the range $\{1, \ldots, n-r\}$ and the index α, runs over the range $\{1, \ldots, \bar{n}-\bar{r}\}$. Now, we introduce a pseudo-framed structure on a product manifold $M\times\overline{M}$ as follows.

For a vector field $(X_p, \overline{X}_{\bar{p}})$ of the product manifold $M\times\overline{M}$ at a point $(p, \bar{p})$, we shall denote $X_p+\overline{X}_{\bar{p}}$. We identify $X\in TM$ with $\tilde{X}\in T(M\times\overline{M})$ by

(2.1) $$\tilde{X}_{(p,\bar{p})}=(X_p, 0_{\bar{p}})=X_p+0_{\bar{p}},$$

where $0_{\bar{p}}$ is the zero vector of M at $\bar{p}$. If $\pi : M\times\overline{M}\to M$ and $\bar{\pi} : M\times\overline{M}\to\overline{M}$ are projections $\pi(p, \bar{p})=p$ and $\bar{\pi}(p, \bar{p})=\bar{p}$, respectively, then $\pi_*\tilde{X}\in T(M\times\overline{M})$ by

(2.2) $$\tilde{X}_{(p,\bar{p})}=(0_p, \overline{X}_{\bar{p}})=0_p+\overline{X}_{\bar{p}}.$$

Differentiable 1-forms on M and $\overline{M}$ are identified with 1-forms on $M\times\overline{M}$ in the same way. If w and $\bar{w}$ are 1-forms on M and $\overline{M}$, respectively, then a 1-form $\tilde{w}$ is defined on $M\times\overline{M}$ by

(2.3) $$\tilde{w}_{(p,\bar{p})}\ (X_p, \overline{X}_{\bar{p}})=w_p(X_p)+\bar{w}_{\bar{p}}(\overline{X}_{\bar{p}}).$$

Now, for any vector fields $X\in TM_p$ and $\overline{X}\in T\overline{M}_{\bar{p}}$, if we put

(2.4) $$F(X, \overline{X})=(fX, f\overline{X}),$$

then F defines a linear map of tangent space $T(M\times\overline{M})$ onto itself. From the last equation, we get

(2.5) $$F^2=(I, \bar{I})-(\eta_x\otimes\xi_x, 0)-(0, \bar{\eta}_\alpha\otimes\bar{\xi}_\alpha),$$

where I and $\bar{I}$ are identity tensor fields of M and $\overline{M}$, respectively. From (2.5) we get

(2.6) $$F^3-F=0,$$

and F has rank $r+\bar{r}$. If we put

$$E_x=(\xi_x, 0) \qquad E_{n-r+\alpha}=(0, \bar{\xi}_\alpha),$$
$$w_x=(\eta_x, 0), \qquad w_{n-r+\beta}=(0, \bar{\eta}_\beta),$$

from which

$$w_x(E_y)=(\eta_x(\xi_y),0),\quad w_{n-r+\alpha}(E_{n-r+\beta})=(0,\bar{\xi}_\alpha(\bar{\eta}_\beta)).$$

Then (2.5) can be written by

$$F^2=\tilde{I}-w_A\otimes E_A, \tag{2.7}$$

where $\tilde{I}=(I,\bar{I})$ and $A,B=1,2,\ldots,\ n+\bar{n}-r-\bar{r}$.

Moreover we get

$$FE_A=0,\qquad w_A\circ F=0,\qquad w_A(E_B)=\delta_{AB}. \tag{2.8}$$

Thus we have

THEOREM 2.1. *Let $M(f,\xi_x,\eta_x)$ and $\bar{M}(\bar{f},\bar{\xi}_\alpha,\bar{\eta}_\alpha)$ be pseudo-framed manifolds of ranks r and $\bar{r}$, respectively. Then the product manifold $M\times\bar{M}$ carries a pseudo-framed structure (F,E_A,w_A) of rank $r+\bar{r}$.*

Let R^m be an m-dimensional Euclidean space. Then R^m has a trivial pseudo-framed structure $(0,d/dt^\alpha,dt^\alpha)$. Hence by Theorem 2.1 we can introduce a pseudo-framed structure on $M\times R^m$ given by

$$F(X,\lambda^\alpha d/dt^\alpha)=(fX,0), \tag{2.9}$$

where λ^α are real valued functions on R^m. Then we have

$$F^2=(I,\bar{I})-(\eta_x\otimes\xi_x,0)-(0,dt^\alpha\otimes d/dt^\alpha).$$

Thus we have

COROLLARY 2.2. *Let $M(f,\xi_x,\eta_x)$ be a pseudo-framed manifold of rank r and R^m an m-dimensional Euclidean space with trivial pseudo-framed structure $(0,d/dt^\alpha,dt^\alpha)$. Then the product manifold $M\times R^m$ has a pseudo-framed structure $(F,\xi_x,d/dt^\alpha,\eta_x,dt^\alpha)$ of rank r given by (2.9).*

Let $M(f,\xi_x,\eta_x)$ and $\bar{M}(\bar{f},\bar{\xi}_x,\bar{\eta}_x)$ be two pseudo-framed manifolds of dimensions $n,\bar{n}$ and ranks $r,\bar{r}$, respectively, where we assume that $n-r=\bar{n}-\bar{r}$. For any vector fields $X_p\in TM_p$ and $\bar{X}_{\bar{p}}\in T\bar{M}_{\bar{p}}$, we define a linear map J of tangent space $T(M\times\bar{M})_{(p,\bar{p})}$ onto itself by

$$J(X,\bar{X})=(fX+\bar{\eta}_x(\bar{X})\xi_x,\ \bar{f}\bar{X}+\eta_x(X)\bar{\xi}_x). \tag{2.10}$$

Then we have

$$J^2=(I,\bar{I}), \tag{2.11}$$

which shows that J is an almost product structure.

Thus we have

THEOREM 2.3. *Let $M(f,\xi_x,\eta_x)$ and $\bar{M}(\bar{f},\bar{\xi}_x,\bar{\eta}_x)$ be two pseudo-framed manifolds. Then the product manifold $M\times\bar{M}$ has an almost product structure J defined by* (2.10).

Now, since R^{n-r} has a trivial pseudo-framed structure $(0,d/dt^x,dt^x)$, (t^x) being the coordinate in R^{n-r}, we can introduce an almost product structure J on a product manifold $M\times R^{n-r}$. If we put

(2.12) $$J(X, \overset{x}{\lambda} d/dt^x) = (fX + \overset{x}{\lambda}\xi_x, \eta_x(X) d/dt^x),$$
then we have $J^2 = (I, \bar{I})$.

Thus we have

THEOREM 2.4. *Let $M(f, \xi_x, \eta_x)$ be a pseudo-framed manifold of rank r. Then the product manifold $M \times R^{n-r}$ has an almost product structure J defined by* (2.12).

Finally, we prove the following:

THEOREM 2.5. *Let $M(f, \xi_x, \eta_x)$ be a pseudo-framed manifold of rank r. If the induced almost product structure J on $M \times M$ is integrable, then the pseudo-framed structure f is integrable.*

Proof. For any vector fields X and Y on $M \times M$, we define an induced almost product structure J on $M \times M$ as follows:

(2.13) $$J(X, Y) = (fX + \eta_x(Y)\xi_x, fY + \eta_x(X)\xi_x).$$

Then the integrability condition of the induced almost product structure J on $M \times M$ is given by

$$[J(X_1+X_2), J(Y_1+Y_2)] - J[J(X_1+X_2), Y_1+Y_2] - J[X_1+X_2, J(Y_1+Y_2)] + [X_1+X_2, Y_1+Y_2] = 0,$$

for any vector fields $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ on $M \times M$. By a direct computation we see that the above condition is equivalent to the following:

(2.14) $$[f,f](X_1, Y_1) + [fX_1, \eta_x(Y_2)\xi_x] - f[X_1, \eta_x(Y_2)\xi_x] + [\eta_x(X_2)\xi_x, fY_1] - f[\eta_x(X_2)\xi_x, Y_1] - \eta_x([fX_2, Y_2] + [X_2, fY_2])\xi_x - \eta_x([\eta_y(X_1)\xi_y, Y_2] + [X_2, \eta_y(Y_1)\xi_y])\xi_x + [\eta_x(X_2)\xi_x, \eta_y(Y_2)\xi_y] = 0,$$

(2.15) $$[f,f](X_2, Y_2) + [fX_2, \eta_x(Y_1)\xi_x] - f[X_2, \eta_x(Y_1)\xi_x] + [\eta_x(X_1)\xi_x, fY_2] - f[\eta_x(X_1)\xi_x, Y_2] - \eta_x([fX_1, Y_1] + [X_1, fY_1])\xi_x - \eta_x([\eta_y(X_2)\xi_y, Y_1] + [X_1, \eta_y(Y_2)\xi_y]) + [\eta_x(X_1)\xi_x, \eta_y(Y_1)\xi_y] = 0.$$

Now, putting $X_2 = Y_2 = 0$ in (2.14) and (2.15) we obtain

(2.16) $$[f,f](X_1, Y_1) = 0,$$

(2.17) $$\eta_x([fX_1, Y_1] + [X_1, fY_1])\xi_{\bar{x}}, -[\eta_x(X_1)\xi_x, \eta_y(Y_1)\xi_y] = 0.$$

Again putting $X_1 = \xi_y$ and $Y_1 = \xi_z$ in (2.17), we get

(2.18) $$[\xi_y, \xi_z] = 0.$$

Putting $Y_1 = \xi_x$ in (2.16), we get

(2.19) $$f[X_1, \xi_x] = [fX_1, \xi_x].$$

Taking account of (2.18), (2.17) can be written by

(2.20) $$\eta_x([fX_1, Y_1] + [X_1, fY_1]) = 0.$$

Using (2.18), (2.19) and (2.20), the integrability conditions (2.14) and (2.15) are expressed as follows:

(2.21) $$[f, f](X_1, Y_1) - \eta_x([\eta_y(X_1)\xi_y, Y_2] + [X_2, \eta_y(Y_1)\xi_y])\xi_x = 0,$$

(2.22) $[f,f](X_2,Y_2)-\eta_x([\eta_y(X_2)\xi_y,Y_1]+[X_1,\eta_y(Y_2)\xi_y])\xi_x=0.$

Again putting $X_1=\xi_z$ and $Y_1=\xi_u$ in (2.21), we get

(2.23) $$\eta_x([\xi_z,Y_2]+[X_2,\xi_u])=0.$$

Similarly we obtain

(2.24) $$\eta_x([\xi_z,Y_1]+[X_1,\xi_u])=0.$$

Then (2.21) and (2.22) are written by

$$[f,f](X_1,Y_1)=0,\qquad [f,f](X_2,Y_2)=0,$$

which shows that the pseudo-framed structure f is integrable.

3. Normal pseudo-framed structure

In the previous section, we have seen that the induced almost product structure J on $M\times \overset{n-r}{R}$ is defined by

(3.1) $$J(X,\overset{x}{\lambda}d/dt^x)=(fX+\overset{x}{\lambda}\xi_x,\ \eta_x(X)d/dt^x)$$

for any vector field X on M and real-valued functions $\overset{x}{\lambda}$ on R^{n-r}. We shall consider the case that the induced almost product structure J is integrable.

DEFINITION. If the induced almost product structure J on $M\times \overset{n-r}{R}$ is integrable, we say that the pseudo-framed structure f on M is normal.

Denoting by $N^A{}_{BC}$ the components of the Nijenhuis tensor $[J,J]$ (X,Y), $N^A{}_{BC}$ is given by

$$N^A{}_{BC}=J^E{}_B\partial_E J_C{}^A-J^E{}_C\partial_E J_B{}^A-J_E{}^A(\partial_B J^E{}_C-\partial_C J^E{}_B),$$

where the indicies $A,B,C,\cdots$, run over the range $\{1,2,\cdots,2n-r\}$.

Considering the Nijenhuis tensor $[J,J]$ of J, they computed

$$[J,J](X+O,Y+O),\quad [J,J](X+O,O+d/dt^x)$$

and

$$[J,J](O+d/dt^x,O+d/dt^y)$$

which rise to five tensors given by

(3.2) $$\begin{aligned}
&N^1(X,Y)=N^i{}_{jk}=[f,f](X,Y)+d\eta_x(X,Y)\xi_x,\\
&N^2(X,Y)=N^x{}_{jk}=(L_{fX}\eta_x)(Y)-(L_{fY}\eta_x)(X),\\
&N^3(X,U)=N^i{}_{jx}=(L_{\xi_x}f)X,\\
&N^4(X,U)=N^x{}_{jy}=-(L_{\xi_x}\eta_y)(X),\\
&N^5(U,V)=N^i{}_{xy}=L_{\xi_x}\xi_y,
\end{aligned}$$

for any vector fields X and Y on M and U,V on $\overset{n-r}{R}$, where L_X denotes the Lie derivative with respect to X. The pseudo-framed structure (f,ξ_x,η_x) is normal if and only if $N^1=0$, that is,

(3.3) $$N^1(X,Y)=[f,f]\ (X,Y)+d\eta_x(X,Y)\xi_x=0.$$

We see that the trivial pseudo-framed structure $(O,d/dt^x,dt^x)$ is normal. Now, we prove the following.

THEOREM 3.1. *Let M and $\overline{M}$ be manifolds with normal pseudo-framed* *structures. Then the pseudo-framed structure of the product manifold* $M\times\bar{M}$ *is normal.*

Proof. Let $M(f,\xi_x,\eta_x)$ and $\bar{M}(\bar{f},\bar{\xi}_\alpha,\bar{\eta}_\alpha)$ be pseudo-framed manifolds of ranks r and $\bar{r}$, respectively. By Theorem 2.1 $M\times\bar{M}$ carries a pseudo-framed structure of rank $r+\bar{r}$ given by (2.4.). Then we compute

$$\begin{aligned}[F,F](X+\bar{X},Y+\bar{Y}) &= [F(X+\bar{X}),F(Y+\bar{Y})]-F[F(X+\bar{X}),Y+\bar{Y}]\\ &\quad -F[X+\bar{X},F(Y+\bar{Y})]+F^2[X+\bar{X},Y+\bar{Y}]\\ &=[(fX+\bar{f}\bar{X},fY+\bar{f}\bar{Y}]-F[fX+\bar{f}\bar{X},Y+\bar{Y}]\\ &\quad -F[X+\bar{X},fY+\bar{f}\bar{Y}]+F^2[X+\bar{X},Y+\bar{Y}]\\ &=([fX,fY],[\bar{f}\bar{X},\bar{f}\bar{Y}])-(f[fX,Y],\bar{f}[\bar{f}\bar{X},\bar{Y}])\\ &\quad -(f[X,fY],\bar{f}[\bar{X},\bar{f}\bar{Y}])+(f^2[X,Y],\bar{f}^2[\bar{X},\bar{Y}]),\\ &=([f,f](X,Y),\ [\bar{f},\bar{f}](\bar{X},\bar{Y}))\end{aligned}$$

from which

$$(3.4)\qquad [F,F]=([f,f],[\bar{f},\bar{f}]).$$

Moreover

$$\begin{aligned}dw_A(X+\bar{X},Y+\bar{Y})E_A &= \{(X+\bar{X})w_A(Y+\bar{Y})-(Y+\bar{Y})w_A(X+\bar{X})\\ &\quad -w_A([X+\bar{X},Y+\bar{Y}])\}E_A\\ &=(X\eta_x(Y)-Y\eta_x(X)-\eta_x([X,Y])\xi_x\\ &\quad +(\bar{X}\bar{\eta}_\alpha(\bar{Y})-\bar{Y}\bar{\eta}_x(\bar{X})-\bar{\eta}_x([\bar{X},\bar{Y}])\bar{\xi}_\alpha,\end{aligned}$$

from which

$$(3.5)\qquad dw_A\otimes E_A=(d\eta_x\otimes\xi_x,\ d\bar{\eta}_\alpha\otimes\bar{\xi}_\alpha).$$

From (3.4) and (3.5) we get

$$(3.6)\qquad N^1(F)=(N^1(f),N^1(\bar{f})),$$

which shows that $M\times\bar{M}$ has a normal pseudo-framed strucutre.

LEMMA 3.2. *If a pseudo-framed structure* (f,ξ_x,η_x) *is normal on* M, *then we have*

(1) $d\eta_x(X,\xi_y)=0$,

(2) $[\xi_x,\xi_y]=0$,

(3) $f[X,\xi_x]=[fX,\xi_x]$,

(4) $d\eta_x(fX,Y)-d\eta_x(X,fY)=0$.

Proof. Putting $Y=\xi_y$ in (3.3), we get

$$(3.7)\qquad -f[fX,\xi_y]+f^2[X,\xi_y]+d\eta_z(X,\xi_y)\xi_z=0.$$

Taking the inner product of the left hand side of the equation by ξ_x, we obtain

$$(3.8)\qquad d\eta_x(X,\xi_y)=0.$$

Secondly, Putting $X=\xi_x$ and $Y=\xi_y$ in (3.3), and using (3.8) we get

$$(3.9)\qquad [\xi_x,\xi_y]=0.$$

Thirdly, from (3.7) and (3.8) we get

(3.10) $$f[X,\xi_y]=f^2[fX,\xi_y]=[fX,\xi_y]-\eta_x([fX,\xi_y])\xi_x=[fX,\xi_y],$$
with the help of (3.8). Fourthly, Putting $Y=fY$ in (3.3), we get
$$[fX,f^2Y]-f[fX,fY]-f[X,f^2Y]+f^2[X,fY]+d\eta_y(X,fZ)\xi_y=0,$$
from which, taking the inner product of the last equation by ξ_x
$$\eta_x([fX,f^2Y])+d\eta_x(X,fY)=0,$$
or

(3.11) $$\eta_x([fX,Y])-fX(\eta_x(Y))+d\eta_x(X,fY)=0.$$

On the other hand, by the definition of $d\eta_x$ we get

(3.12) $$fX(\eta_x(Y))-Y(\eta_x(fX))-\eta_x([fX,Y])-d\eta_x(fX,Y)=0.$$

Adding the last two equations we have

(3.13) $$d\eta_x(X,fY)-d\eta_x(fX,Y)=0.$$

By the definition of Lie derivative, (1), (2), (3) and (4) are equivalent to $N^2=0$, $N^5=0$, $N^3=0$ and $N^2=0$, respectively.

Thus we have also the following (cf. [11]): If a pseudo-framed structure is normal, that is, $N^1=0$, then we have
$$N^2=N^3=N^4=N^5=0.$$

Finally, we prove the following.

THEOREM 3.3. *Let $M(f,\xi_x,\eta_x)$ be a manifold with normal pseudo-framed structure of rank r. If f and η are Killing tensors, the structure tensors f, ξ_x and η_x are covariantly constant, that is,*
$$\nabla_X f=0,\quad \nabla_X\xi_x=0,\quad \nabla_X\eta_x=0.$$

Proof. Since η_x are Killing forms we get
$$(\nabla_X\eta_x)(Y)+(\nabla_Y\eta_x)(X)=0,$$
from which

(3.14) $$d\eta_x(X,Y)=-2(\nabla_Y\eta_x)(X).$$

By the normality N^3 vanishes identically, that is, $L_{\xi x}f=0$, and hence we get
$$(L_{\xi_x}F)(X,Y)=(L_{\xi_x}g)(X,fY)=0,$$
from which
$$(\nabla_{\xi_x}F)(X,Y)=(\nabla_X F)(Y,\xi_x)+(\nabla_Y F)(X,\xi_x).$$
Since F is a Killing tensor, we get

(3.15) $$(\nabla_{\xi_x}F)(X,Y)=0.$$

Since f is a Killing tensor, by the normality $N^3=0$, we get
$$0=(\nabla_{\xi_x}f)X=-(\nabla_X f)\xi_x=f(\nabla_X\xi_x).$$
Hence if X is orthogonal to ξ_x, then we can put $X=fZ$ for some Z and we obtain
$$d\eta_x(X,Y)=-2g(X,\nabla_Y\xi_x)=-2g(fZ,\nabla_Y\xi_x)=-2g(Z,f(\nabla_Y\xi_x))=0.$$
Thus, from (3.8) we have

(3.16) $$d\eta_x=0.$$

From (3.14) we get

(3.17) $$\nabla_X \eta_x = 0,$$

from which

(3.18) $$\nabla_X \xi_x = 0.$$

On the other hand, by the normality and (3.16) we get

$$(\nabla_{fX} f) Y - (\nabla_{fY} f) X - f(\nabla_X f) Y + f(\nabla_Y f) X = 0.$$

Since f is a Killing tensor, we get

$$-(\nabla_Y f) f X + (\nabla_X f) f Y - 2f(\nabla_X f) Y$$
$$= -(\nabla_Y f^2) X + f(\nabla_Y f) X + (\nabla_X f^2) Y - f(\nabla_X f) Y - 2f(\nabla_X f) Y = 0,$$

from which $f(\nabla_X f) Y = 0$.

Applying f to the last equation, we get

$$(\nabla_X f) Y - \eta_x((\nabla_X f) Y) \xi_x = 0,$$

from which we have

(3.19) $$\nabla_X f = 0.$$

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