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ON δ-CONTINUOUS FUNCTIONS

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1. Introduction

The purpose of the present note is to introduce a new class of functions called δ-continuous and investigate the relationships between δ-continuity and near-compactness due to M. K. Singal and Asha Mathur [9]. The concepts of continuity and δ-continuity are independent of each other and both imply almost-continuity due to M. K. Singal and A. R. Singal [10]. However, an almost-continuous function need not be δ-continuous.

Throughout the present note spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated. A subset S of a space X is said to be *regular open* (resp. *regular closed*) if $\mathrm{Int}(\mathrm{Cl}(S))=S$ (resp. $\mathrm{Cl}(\mathrm{Int}(S))=S$), where $\mathrm{Cl}(S)$ (resp. $\mathrm{Int}(S)$) denotes the closure (resp. interior) of S. A point $x\in X$ is said to be *δ-cluster* point of S [12] if $S\cap U\neq\phi$ for every regular open set U containing x. The set of all δ-cluster points of S are called the *δ-closure* of S and denoted by $[S]_\delta$. If $[S]_\delta=S$, then S is called *δ-closed*. The complement of a δ-closed set is called *δ-open*. For basic properties of δ-closed sets, refer to [4] and [12].

2. Characterizations

DEFINITION 2.1. A function $f: X\to Y$ is said to be *δ-continuous* if for each $x\in X$ and each open neighborhood V of $f(x)$, there exists an open neighborhood U of x such that $f(\mathrm{Int}(\mathrm{Cl}(U)))\subset\mathrm{Int}(\mathrm{Cl}(V))$.

We denote the semi-regularization of a space X by X_s and define a function $f_s: X_s\to Y_s$ associated with a function $f: X\to Y$ as follows: $f_s(x)=f(x)$ for each $x\in X_s$. The following theorems are easy consequences of the above definition and the proofs are thus omitted.

THEOREM 2.2. *For a function $f: X\to Y$, the following are equivalent:*

(1) *f is δ-continuous.*

(2) *For each $x\in X$ and each regular open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U)\subset V$.*

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(3) $f([A]_\delta) \subset [f(A)]_\delta$ *for every* $A \subset X$.
(4) $[f^{-1}(B)]_\delta \subset f^{-1}([B]_\delta)$ *for every* $B \subset Y$.
(5) *For every regular closed set* F *of* Y, $f^{-1}(F)$ *is* δ*-closed in* X.
(6) *For every* δ*-closed set* F *of* Y, $f^{-1}(F)$ *is* δ*-closed in* X.
(7) *For every* δ*-open set* V *of* Y, $f^{-1}(V)$ *is* δ*-open in* X.
(8) *For every regular open set* V *of* Y, $f^{-1}(V)$ *is* δ*-open in* X.

THEOREM 2.3. *A function* $f: X \to Y$ *is* δ*-continuous if and only if* $f(\mathfrak{F})$ δ*-converges to* $f(x)$ *for each* $x \in X$ *and each filter base* $\mathfrak{F}$ δ*-converging to* x.

THEOREM 2.4. *A function* $f: X \to Y$ *is* δ*-continuous if and only if* $\{f(x_\alpha)\}_{\alpha \in D}$ δ*-converges to* $f(x)$ *for each* $x \in X$ *and each net* $\{x_\alpha\}_{\alpha \in D}$ δ*-converging to* x.

THEOREM 2.5. *A function* $f: X \to Y$ *is* δ*-continuous if and only if* $f_s: X_s \to Y_s$ *is continuous.*

3. Basic properties*)

The following properties are easily obtained and the proofs are thus omitted.

THEOREM 3.1. *If* $f: X \to Y$ *and* $g: Y \to Z$ *are* δ*-continuous, then so is* $g \circ f: X \to Z$.

THEOREM 3.2. *For a function* $f: X \to Y$, *the following are true*:
(1) *If* f *is* δ*-continuous and* X_0 *is open in* X, *then* $f|X_0: X_0 \to Y$ *is* δ*-continuous.*
(2) *If* $\{U_\alpha | \alpha \in \mathcal{A}\}$ *is a cover of* X *by regular open sets and* $f|U_\alpha: U_\alpha \to Y$ *is* δ*-continuous for each* $\alpha \in \mathcal{A}$, *then* f *is* δ*-continuous.*

THEOREM 3.3. *Let* $f_\alpha: X_\alpha \to Y_\alpha$ *be a function for each* $\alpha \in \mathcal{A}$ *and* $f: \prod X_\alpha \to \prod Y_\alpha$ *a function defined by* $f(\{x_\alpha\}) = \{f_\alpha(x_\alpha)\}$ *for each point* $\{x_\alpha\} \in \prod X_\alpha$. *Then,* f *is* δ*-continuous if and only if* f_α *is* δ*-continuous for each* $\alpha \in \mathcal{A}$.

THEOREM 3.4. *A function* $f: X \to \prod X_\alpha$ *is* δ*-continuous if and only if* $p_\beta \circ f$ *is* δ*-continuous for each* $\beta \in \mathcal{A}$, *where* p_β *is the projection of* $\prod X_\alpha$ *onto* X_β.

COROLLARY 3.5. *Let* $f: X \to Y$ *be a function and let* $g: X \to X \times Y$, *given by* $g(x) = (x, f(x))$, *be its graph function. Then* f *is* δ*-continuous if and only if* g *is* δ*-continuous.*

4. Comparisons

DEFINITION 4.1. A function $f: X \to Y$ is said to be *almost-continuous* [10] (resp. *θ-continuous* [2], *strongly θ-continuous*) if for each $x \in X$ and each

*)The author is grateful to the referee for his valuable suggestions to improve the original form of this section.

open neighborhood V of $f(x)$, there exists an open neighborhood U of x such thet $f(U)\subset \text{Int}(\text{Cl}(V))$ [resp. $f(\text{Cl}(U))\subset \text{Cl}(V)$, $f(\text{Cl}(U))\subset V$].

DEFINITION 4.2. A function $f: X\to Y$ is said to be *almost-open* [10] if for each regular open set U of X, $f(U)$ is open in Y.

THEOREM 4.3. (1) *If $f: X\to Y$ is strongly θ-continuos and $g: Y\to Z$ is almost-continuous, then $g\circ f: X\to Z$ is δ-continuous.*

(2) *The following implications hold:*
strongly θ-continuous $\Rightarrow$ δ-continuous $\Rightarrow$ almost-continuous.

Proof. These are immediate consequences of the definitions.

The following two examples show that the concepts of δ-continuity and continuity are independent of each other and that none of implications in (2) of Theorem 4.3 can be reversible.

EXAMPLE 4.4. Let X be the real numbers with the usual topology, Y the real numbers with the co-countable topology and $f: X\to Y$ be the identity function. Then f is δ-continuous but not continuous.

EXAMPLE 4.5. Let $X=Y=\{a, b, c\}$, $\mathfrak{T}_X=\{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\}$ and $\mathfrak{T}_Y=\{\phi, \{a\}, \{c\}, \{a, c\}, Y\}$. Let $f: (X, \mathfrak{T}_X)\to(Y, \mathfrak{T}_Y)$ be the identity function. Then f is continuous but not δ-continuous.

THEOREM 4.6. *For a function $f: X\to Y$, the following are true:*

(1) *If Y is semi-regular and f is δ-continuous, then f is continuous.*

(2) *If X is semi-regular and f is almost-continuous, then f is δ-continuous.*

Example 4.4 (resp. Example 4.5) shows that in (1) (resp. (2)) of Theorem 4.6, semi-regularity on Y (resp. X) can not be dropt.

COROLLARY 4.7. *If X and Y are semi-regular spaces, then the following concepts on a function $f: X\to Y$: δ-continuity, continuity and almost-continuity are equivalent.*

DEFINITION 4.8. A space X is said to be *almost-regular* [8] if for each regular closed set $F\subset X$ and each $x\notin F$, there exist disjoint open sets U and V in X such that $x\in U$ and $F\subset V$.

In [8], it has been known that almost-regularity strictly weaker than regularity and is independent to semi-regularity, however, every almost-regular and semi-regular space is regular.

THEOREM 4.9. *For a function $f: X\to Y$, the following are true:*

(1) *If Y is almost-regular and f is θ-continuous, then f is δ-continuous.*

(2) *If X is almost-regular, Y is semi-regular and f is δ-continuous, then f is strongly θ-continuous.*

Proof. (1) This follows easily from Theorem 2.2 and the fact that a space Y is almost-regular if and only if for each $y \in Y$ and each regular open set V containing y there exists a regular open set V_0 such that $y \in V_0 \subset \mathrm{Cl}(V_0) \subset V$ [8, Theorem 2.2].

(2) Let $x \in X$ and V be an open neighborhood of $f(x)$. There exist regular open sets $V_0 \subset Y$ and $U_0 \subset X$ such that $x \in U_0$ and $f(U_0) \subset V_0 \subset V$. Moreover, there exists an open set $U \subset X$ such that $x \in U \subset \mathrm{Cl}(U) \subset U_0$; hence $f(\mathrm{Cl}(U)) \subset V$. This shows that f is strongly θ-continuous.

Example 4.5 (resp. Example 4.4) shows that in (1) (resp. (2)) of Theorem 4.9, almost-regularity (resp. semi-regularity) on Y can not be dropt. Since every almost-continuous function is θ-continuous [5, Lemma 6], from Theorem 4.6 and Theorem 4.9 we have

COROLLARY 4.10. *If X and Y are regular spaces, then the following concepts on a function $f: X \to Y$: θ-continuity, almost-continuity, δ-continuity, continuity and strongly θ-continuity are equivalent.*

THEOREM 4.11. *If a function $f: X \to Y$ is θ-continuous and almost-open, then it is δ-continuous.*

Proof. Let $x \in X$ and V be an open neighborhood of $f(x)$. There exists an open neighborhood U of x such that $f(\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$; therefore, $f(\mathrm{Int}(\mathrm{Cl}(U))) \subset \mathrm{Cl}(V)$. Since f is almost-open, we have $f(\mathrm{Int}(\mathrm{Cl}(V))) \subset \mathrm{Int}(\mathrm{Cl}(V))$. This shows that f is δ-continuous.

5. Nearly-compact spaces

DEFINITION 5.1. The graph $G(f)$ of a function $f: X \to Y$ is said to be *δ-closed* if $G(f)$ is δ-closed in the product space $X \times Y$.

In [11], T. Thompson defined $G(f)$ to be *r-closed* if for each $(x, y) \notin G(f)$, there exist regular open sets $U \subset X$ and $V \subset Y$ containing x and y, respectively, such that $f(U) \cap V = \phi$. By a straightforward calculation, we have

THEOREM 5.2. *For a function $f: X \to Y$, the following are true:*

(1) *$G(f)$ is δ-closed if and only if $G(f)$ is r-closed.*

(2) *If f is δ-continuous and Y is Hausdorff, then $G(f)$ is δ-closed.*

DEFINITION 5.3. A subset K of a space X is said to be *N-closed relative to* X [1] if every cover of K by regular open sets in X has a finite subcover. A space X is said to be *nearly-compact* [9] if X is N-closed relative to X.

THEOREM 5.4. *Let $f: X \rightarrow Y$ be a function with a δ-closed graph. If K is N-closed relative to Y* (resp. X), *then $f^{-1}(K)$* (resp. $f(K)$) *is δ-closed in X* (resp. Y).

Proof. We prove only the first case, the proof of the second being analogous. Suppose that K is N-closed relative to Y. For each $x \notin f^{-1}(K)$ and each $y \in K$, $(x, y) \notin G(f)$ and hence, by Theorem 5.2, there exist regular open sets $U(y) \subset X$ and $V(y) \subset Y$ containing x and y, respectively, such that $f(U(y)) \cap V(y) = \phi$. Therefore, there exists a finite subset $K_0 \subset K$ such that $K \subset \cup \{V(y) \mid y \in K_0\}$. Put $U(x) = \cap \{U(y) \mid y \in K_0\}$, then $U(x)$ is a regular open set containing x and $U(x) \cap f^{-1}(K) = \phi$. This shows that $x \notin [f^{-1}(K)]_\delta$ and hence $f^{-1}(K)$ is δ-closed.

THEOREM 5.5. *If Y is a nearly-compact space and a function $f: X \rightarrow Y$ has a δ-closed graph, then f is δ-continuous.*

Proof. Let F be any regular closed set of Y. Since Y is nearly-compact, F is N-closed relative to Y [1, Theorem 2.6] and hence $f^{-1}(F)$ is δ-closed in X by Theorem 5.4. This shows that f is δ-continuous.

COROLLARY 5.6 (Thompson [11]). *Let $f: X \rightarrow Y$ be a function with an r-closed graph. If Y is nearly-compact, then f is almost-continuous.*

Proof. This follows immediately from Theorem 5.2 and Theorem 5.5.

LEMMA 5.7. *If $f: X \rightarrow Y$ is a δ-continuous function and K is N-closed relative to X, then $f(K)$ is N-closed relative to Y.*

Proof. This follows immediately from Theorem 2.5 and the following result: A subset K of a space X is N-closed relative to X if and only if K is compact in X_S [6, Theorem 3.1].

THEOREM 5.8. *Near-compactness is preserved under δ-continuous surjections.*

COROLLARY 5.9 (Singal and Mathur [9]). *Near-compactness is preserved under almost-continuous and almost-open surjections.*

Proof. Every almost-continuous function is θ-continuous [5, Lemma 6]. Therefore, this is an immediate consequence of Theorem 4.11 and Theorem 5.8.

DEFINITION 5.10. A function $f: X \rightarrow Y$ is said to be *δ-perfect* [7] if for every filter base $\mathfrak{F}$ in $f(X)$ δ-converging to $y \in Y$, $f^{-1}(\mathfrak{F})$ is δ-directed toward $f^{-1}(y)$.

THEOREM 5.11. *Let X be a nearly-compact space and Y a Hausdorff space. If $f: X \rightarrow Y$ is a δ-continuous function, then it is δ-perfect.*

Proof. By Theorem 4.1 of [1] and Theorem 2.5, f_s is a continuous function of a compact space X_s into a Hausdorff space Y_s. Therefore, f_s is a closed function with compact point inverses and hence f is δ-perfect [7, Corollary 3.6].

THEOREM 5.12. *Let X be a Hausdorff space and Y a nearly-compact space. If $f: X \to Y$ is a δ-perfect function, then it is δ-continuous.*

Proof. Let F be a regular set of Y. By Theorem 2.6 of [1], F is N-closed relative to Y and hence so is $f^{-1}(F)$ [7, Theorem 3.4]. Since X is Hausdorff, $f^{-1}(F)$ is δ-closed and hence f is δ-continuous by Theorem 2.2.

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