J. Korean Math. Soc.
Vol. 16, No. 2, 1980

ON THE STRUCTURE OF HENSEL FIELDS

BY KUKJIN KIM

0. Introduction

In this paper, we shall see the entire generalization of the results by Ax and Kochen ([1], [2]). The proof the author shall take in this paper is closely modeled to Theorem 13, 14 and 15 in [2], so to the model-completeness of algebraically closed fields with valuation by A. Robinson [5]. But in the theory of Hensel fields, we have to see that the most essential conception characterizing Hensel fields for both characteristics is algebraic completeness (see p. 182 in [3] by J. Ax). Furthermore, the extraction-packing method in [4] introduced by the author shall give similar results in valued fields. In this paper, the author has restricted all arguments by valuation theory, so if one is familiar with the arguments by Ax and Kochen ([1], [2]), the proof would be almost trivial.

The main results are the following:

(1) *The theory L_V of Hensel fields V is model-complete if the theory $L_{\bar{V}}$ of the residue class fields $\bar{V}$ is model-complete in such sense that all quantifiers range over V.*

(2) *All Hensel fields are elementarily equivalent if the residue class fields are elementarily equivalent. Thus we also have that*

(3) *$GF(p)((t))$ is decidable.*

NOTATIONS

$K((t))$: formal power series fields over fields K.

$K \equiv K'$: two fields K and K' are elementarily equivalent.

$Ch(V)$: the characteristic of a valued field V.

$\bar{K}$: the algebraic closure of a field K.

$K[t] = \left\{ \sum_{i=1}^{n} f_i t^i \mid f_i \in K \right\}$.

$K(t) = \{ab^{-1} \mid a, b \in K[t]\}$.

$K(t^G) = \{ab^{-1} \mid a, b \in K[t^G]\}$, where $K[t^G] = \left\{ \sum_{i=1}^{n} f_i t^{g_i} \mid g_i \in G \right\}$.

Received August 20, 1979

1. Hensel fields

We fiirstly define Hensel fields in a countable set of elementary statements for both characteristics but we have to descriminate the formulations according to $Ch(V)$ of valued fields V.

Case 1, $Ch(V)=0$.

A valued field V is called to be a Hensel field if

1. *Hensel's lemma holds in V.*
2. *ord(V) = a Z-group G.*
3. *There is a prime element t such that ord(t) = 1.*

Case 2, $Ch(V)\neq 0$.

1. *Hensel's lemma holds in V.*
2. *ord(V) = a Z-group G.*
3. *There is a prime element t in V such that ord(t) = 1.*
4. $\forall c_1 \cdots\cdots c_{p-1} \in V \exists y \notin V[y^p + c_1 y^{p-1} + \cdots\cdots\cdots + c_{p-1} = 0]$

and

$$\forall y \notin V \exists c_1 \cdots\cdots c_{p-1} \in V[y^p + c_1 y^{p-1} + \cdots\cdots\cdots + c_{p-1} = 0].$$

5. $\forall x \notin \overline{V(x)} \supsetneqq \overline{V}$ *or* $ord(V(x)) \supsetneqq ord(V)]$.

So all quantifiers range over $V_p \supseteq V$ *and we call the corresponding elements to V in* V_p *to be a Hensel field.*

THEOREM 1.1 *Let the theories of* $\overline{V}$, V *be* $L_V, L_{\overline{V}}$. *If* $L_{\overline{V}}$ *is model-complete,* L_V *is also model-complete. But when* $Ch(V)\neq 0$, *all quantifiers range over V.*

Proof. We divide the proof into two cases for $Ch(V)=0$ and $Ch(V)\neq 0$. But before going to make the proof, we have to keep a couple of basics required in our mind.

1. The theory of Z-groups is model-complete by A. Robinson and E. Zakon [6].

2. For a pair of Hensel fields V, V' such that $V\subseteq V'$, if one takes any element $c\in V'$ but V, c is always transcendental over V. Because suppose that c is algebraic over V, by Proposition 15 in [3], then we can say that $[V(c):V]=p$ if c is not algebraically complete to V (see p. 182 in [3]). So we can always assume that c is algebraically complete to V by the axioms 4, 5. Since $L_{\overline{V}}$ and the theory of Z-groups are model-complete, the algebraic completeness gives a contradiction by the same arguments at Ax and Kochen [1] [2] [3]. Thus the relative algebraic closure $V(c)_H$ of $V(c)$ in V' is a Hensel field.

3. By A. Robinson [5], to see that L_V is model-complete, the proof suffices our aim that for any Hensel field V, V' such that $V\subseteq V'$ and any

existential statesment $\exists xR(x)$ defined in V, $V'|=\exists xR(x)\longrightarrow V|=\exists xR(x)$ where $R(x)$ may contain further existential quantifiers restricted to V'.

4. By the same arguments of the model-completeness of algebraically closed fields with valuation by A. Robinson [5] (or Theorem 15 in [2]), we can assume that the transcendental degree of V' over V is 1. Therefore, there are three cases of V' whose transcendental degree over V is 1.

Case 1. $\bar{V}'=\bar{V}$ and $ord(V')=ord(V)$.
Case 2. $\bar{V}'\supsetneqq\bar{V}$ and $ord(V')=ord(V)$.
Case 3. $\bar{V}'=\bar{V}$ and $ord(V')\supsetneqq ord(V)$.

Case A. $Ch(V)=0$.

Case 1 and case 3 have already been proved in Theorem 15([2]).

Consider case 2, and define the next statements L_2^* for only case 2.

1. V's diagram $D(V)$.
2. L_V's axioms.
3. $c\neq a_i$ where a_i varies over V.
4. $ord(c-a_i)=b_i$ holds in V' where a_i varies over V.
5. $\bar{c}\neq\bar{a}_i$ where a_i varies over V.

Let V_2^* be a model of L_2^*, then it is not diffcult to see that $\bar{V}_2^*$ contains an isomorphic copy of $\bar{V}'$. Let the corresponding element to $\bar{c}$ in $\bar{V}_2^*$ be $\bar{d}$. Then by Theorem 13 in [2], $V(\bar{c})$ and $V(\bar{d})\,(\subseteq V_2^*)$ are value isomorphic[1] and the value isomorphism can be easily extended on the Henselizations $V(\bar{c})_s$ and $V(\bar{d})_s$. Again we have a value isomorphism from $V(\bar{c})_s(\bar{c}_i)$ onto $V(\bar{d})_s(\bar{d}_i)$. Since $\bar{V}_2^*$ contains $\bar{V}'$, V_2^* contains V' as a valued field. Then by the Godel's completeness theorem, there exists a finite set of $a_1,\cdots\cdots,a_k$, $a_{k+1},\cdots\cdots,a_m$ of V and elements $g_{j+1},\cdots\cdots,g_k$ of $ord(V)$ such that

$$\bar{c}\neq\bar{a}_i\wedge\cdots\wedge\bar{c}\neq\bar{a}_m\wedge ord(c-a_{j+1})$$
$$=g_{j+1}\wedge\cdots\wedge ord(c-a_k)=g_k\longrightarrow\exists xR(x)$$

is deducible from $L_V\cup D(V)$.

Since case 2 only occurs when $L_{\bar{V}}$ is a theory of infinite models, the system has a solution in V.

Case B. $Ch(V)\neq 0$.

Consider the next statements L_1^* for only case 1.

1. V's diagram $D(V)$.
2. $L_V=L(V_p, V)$.
3. $c\neq a_i$ where a_i varies over V.
4. $ord(c-a_i)=b_i$ holds in V' where a_i varies over V.

---

1) Assuming a cross-section $x\longrightarrow t^x$, V can be assumed to be a $F(t^G)_s$ by Proposition 17 in [3] where $F=\bar{V}$, $ord(V)=G$. So $V'=F'(t^G)_s$.
Furthermore Theorem 13 and case (i) are still the theorem to case 2.

5. $ord(c) \in G$ and $c \in F$ where $ord(V)=G$ and $\bar{V}=F$.

Since (V'_p, V') is a model of L_1^*, L_1^* is a consistent theory defined in the first order language. Let (V_p^*, V^*) be any model of L_1^* and d be the corresponding element in V^* to the constant c in L_1^*.

Then d is transcendental over $V \subseteq V^*$. So it is trivial that $V(c)$ and $V(d) \subseteq V^*$ are isomorphic as fields. Then it is not difficult to see that $V(c)$ and $V(d)$ are value isomorphic because if for any $\bar{v} \in \bar{V}$, there exists $a \in \bar{V}$ such that $ord(c-a) \geq ord(c-\bar{v})$ for any $\bar{v} \in \bar{V}$, by lemma 1 (see p. 451 in [2]) there exists $v \in V$ such that $ord(H[v])=ord(H[c])$ for any $H[X] \in V[X]$. Then we have the equality

$$ord(H[c])=\max(c \in V)\min(ord(H[v]),\ ord(H[c]-H[v]))$$

So $H[c]-H[v]=(c-v)\ H'\ [c]$ with $deg\ H[X] > deg\ H'[X]$ and it shows that 4 in L_1^* uniquely defines the valuation on $V(c)=V(d)$. Therefore, the proof suffices our aim that for any $\bar{v} \in \bar{V}$, there exists $a \in V$ such that $ord(c-a) \geq ord(c-\bar{v})$. By Proposition 17 in [3], the proof suffices our aim that V is algebraically complete. Thus we have shown that $V(c)$ and $V(d)$ are value isomorphic.

By the Henselizations $V(c)_s$, $V(d)_s$ in Proposition 13([3]), we can define a value isomorphism F_V from $V(c)_s$. Then we can define (extend) a value isomorphism $\bar{F}$ from V' onto $\bar{F}(V') \subseteq V^*$ by $V(c)_s \cong V(d)_s$ as valued fields and $V(c)_s$, $V(d)_s$: Hensel fields. Thus we have shown that for any model (V_p^*, V^*) of L_1^*, there is a value isomorphism $\bar{F}$ from V' into V^*. Since $\exists x R(x)$ is an existential statement which holds in V', $V^* \models \exists x R(x)$. By the Godel's completeness theorem, $\exists x \in V'[R(x)]$ is deducible from L_1^*. In other words, there exists a finite set of elements $a_1, \cdots\cdots, a_j, \cdots\cdots, a_k$ of V and element $g_{j+1}, \cdots\cdots, g_k$ of $ord(V)$ such that

$$\bar{c} \in F \wedge ord(c) \in G \wedge c \neq a_1 \wedge \cdots \wedge c \neq a_j \wedge ord(c-a_{j+1}) = g_{j+1} \wedge \cdots \wedge ord(c-a_k) = g_k \longrightarrow \exists x \in V'[R(x)]$$

is deducible for $L(V_p, V) \cup D(V)$.

Since c is a constant not occured in $L(V_p, V) \cup D(V)$ and $\bar{V}=\bar{V}'$, $ord(V)=ord(V')$,

$$\exists x (x \neq a_1 \wedge \cdots \wedge x \neq a_j \wedge ord(x-a_{j+1}) = g_{j+1} \wedge \cdots \wedge ord(x-a_k) = g_k \longrightarrow \exists x \in V'[R(x)]$$

is deducible from $L(V_p, V) \cup D(V)$.

Therefore, the Theorem for case 1 is satisfied by that the next system has a solution in V under the condition, i.e., the system has a solution in V' $x \neq a_i$, $i=1, 2, \cdots, j$ and $ord(x-a_i)=g_i$, $i=j+1, \cdots, k$.

By the same arguments of the model-completeness of algebraically closed fields with valuation by A. Robinson [5], the system can be reduced to a system $\bar{y}=0$ and $\bar{y}=c_i$ $i=2,\cdots,m$ to have a solution in $\bar{V}$ under the condition that the system has a solution in $\bar{V}'$, we have shown the proof for case 1.

For the case 2, the proof is quite same to the proof in case A.

For the case 3, the proof is essentially same to the proof of Theorem 14, 15 in [2]. *Q.E.D.*

THEOREM 1.2. *Let V, V' be Hensel fields such that $\bar{V}\equiv\bar{V}'$ and $Ch(V)=Ch(V')$, then $V\equiv V'$ as valued fields.*

Proof. Any Hensel field V contains a Hensel field V_z such that $ord(V_z)=Z!$ and $\bar{V}_z=\bar{V}$ as follows, take the prime element t in V and then consider a valued ring $\bar{V}[t]$ contained in V. $\bar{V}[t]$ is also extended to $\bar{V}(t)$ and by the Henselization $\bar{V}(t)_s$ of $\bar{V}(t)$ in V, we have a Hensel field $V_z=\bar{V}(t)_s$ if $Ch(V)=0$. Otherwise, taking the maximal immediate, algebraic extension $\bar{V}(t)_H$ (the relative algebraic closure) of $\bar{V}(t)_s$ in V, we have a Hensel field V_z desired. By Theorem 1.1, $V\equiv V_z$ $V'\equiv V_z'$ as valued fields. Again by Theorem 1.1 and the Cauchy completion, $V_z\equiv V((t))$ and $V_z'\equiv V'((t))$ as valued fields. By saturated models, we have nonprincipal ultrafilters D_m, D_n over cardinals such that $\Pi\bar{V}/D_m\cong\Pi\bar{V}'/D_n$. Since $\Pi\bar{V}/D_m((t))$ and $\Pi\bar{V}'/D_n((t))$ are Hensel fields, again by Theorem 1.1 we have

$$V\equiv\Pi\bar{V}/D_m((t))\cong\Pi\bar{V}'/D_n((t))\equiv V'$$

as valued fields. *Q.E.D.*

THEOREM 1.3 (without the Continuum Hypothesis)

$\{p\in P/Q_p|=C_2(d)\}\in D$, *where P is the set of all positive primes.*

THEOREM 1.4. *If $L_{\bar{V}}$ is a complete theory, L_V is also a complete theory.*

THEOREM 1.5.

1. $GF(p)((t))$ *is decidable.*
2. $\overline{GF(p)}((t))$ *is decidable.*

REMARK

If the readers are familiar with [1], [2], [3] and [4], they may realize that Theorem 1.2 is making the end to the questions of the model theory of local fields[2] but the value group is a Z-group. The detailed arguments omitted in Theorem 1.1 and Theorem 1.2 shall appear in the author's papers "*On the structure of algebraically complete fields*" and "*The decidability of algebraically complete fields*"

2) the conjectures by E. Artin, S. Lang, A. Robinson and R. Robinson.

References

1. J. Ax and S. Kochen, *Diophantine problems over local fields I, II*, Am. J. Math **87**(1965), 605-648.
2. J. Ax and S. Kochen, *Diophantine problems over local fields III*, Ann. of. Math. **83** (1966), 437-456.
3. J. Ax, AMS Symposium, 1973, 161-190.
4. K. J. Kim, *The extraction-packing method*, to appear.
5. A. Robinson, *Complete theories*, 1956, North Holland Pub. Company.
6. A. Robinson and E. Zakon, *Elementary properties of ordered abelian group*, Trans, Am. Math **96**(1960), 222-236.