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SUBMANIFOLDS OF CODIMENSION 3 OF A KAEHLERIAN MANIFOLD (I)

BY SANG-SEUP EUM, U-HANG KI, UN KYU KIM AND YOUNG HO KIM*)

0. Introduction

It is well known that a submanifold of codimension 3 of an Hermitian manifold admits an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced from the almost Hermitian structure of the ambient manifold.

In the present paper we investigate a submanifold of codimension 3 of a $(2n+4)$-dimensional Kaehlerian manifold admitting an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure.

Firstly, we study the structure induced on the submanifold of codimension 3 of a $(2n+4)$-dimensional Kaehlerian manifold. In section 1, we define the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure and we show that this kind of structure gives an almost contact metric structure when $\lambda^2+\mu^2+\nu^2=1$ and we find a necessary and sufficient condition that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure be antinormal. In section 2, we study some equations concerning the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure and we show that in order for the structure to be antinormal, it is necessary and sufficient that h and f anticommute, where h is the second fundamental tensor with respect to the distinguished normal.

Next, we study the submanifold of codimension 3 of a $(2n+4)$-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c. In section 3, we investigate the submanifolds satisfying the condition $\lambda^2+\mu^2+\nu^2=1$ and we show that an umbilical submanifold with respect to the distinguished normal is an intersection of a complex cone and a sphere, that is, such a submanifold is an extended Brieskorn manifold. In section 4, we show that an antinormal minimal submanifold is a submanifold of a $(2n+3)$-dimensional Euclidean space under some conditions. Moreover in this section, we show that a complete submanifold of codimension 3 of a Euclidean space E^{2n+4} is a plane or a ruled surface under some conditions. In section 5, we find a necessary and sufficient condition that the connection induced in the normal bundle of the submanifold to be trivial. Moreover in this section, we study a complete submanifold of codimension 3 of a $(2n+4)$-dimen

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sional Euclidean space E^{2n+4} whose normal connection is flat and characterize this submanifold under some conditions.

1. Structures induced on submanifolds of codimension 3 of an almost Hermitian manifold

Let M^{2n+4} be a $(2n+4)$-dimensional almost Hermitian manifold covered by a system of coordinate neighborhoods $\{U; x^A\}$ and denote by g_{CB} components of the Hermitian metric tensor and by $F_B{}^A$ those of the almost complex structure tensor of M^{2n+4}, where here and in the sequel the indices A, B, C, ... run over the range $1', 2', \ldots, (2n+4)'$. Then we have

(1.1) $$F_C{}^B F_B{}^A = -\delta_C{}^A, \qquad g_{ED} F_C{}^E F_B{}^D = g_{CB},$$

$\delta_C{}^A$ being the Kronecker delta.

Let M^{2n+1} be a $(2n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhood $\{V: y^h\}$ and immersed isometrically in M^{2n+4} by the immersion $i: M^{2n+1} \to M^{2n+4}$, where here and in the sequel the indices $h, i, j, \ldots$ run over the range $1, 2, \ldots, (2n+1)$. We identify $i(M^{2n+1})$ with M^{2n+1} itself and represent the immersion by

(1.2) $$x^A = x^A(y^h).$$

We now put $B_i{}^A = \partial_i x^A$, $(\partial_i = \partial/\partial y^i)$. Then $B_i{}^A$ are $2n+1$ linearly independent vectors of M^{2n+4} tangent to M^{2n+1}. And denote by C^A, D^A, and E^A three mutually orthogonal unit normals to M^{2n+1}. Then denoting by g_{ji} components of the induced metric tensor of M^{2n+1}, we have

(1.3) $$g_{ji} = g_{CD} B_j{}^C B_i{}^D$$

since the immersion is isometric.

As to the transforms of $B_i{}^A, C^A, D^A$, and E^A by $F_B{}^A$, we have respectively the following equations of the form

(1.4) $$F_C{}^A B_i{}^C = f_i{}^h B_h{}^A + u_i C^A + v_i D^A + w_i E^A,$$

(1.5) $$F_B{}^A B^B = -u^h B_h{}^A - \nu D^A + \mu E^A,$$

(1.6) $$F_B{}^A D^B = -v^h B_h{}^A + \nu C^A - \lambda E^A,$$

(1.7) $$F_B{}^A E^B = -w^h B_h{}^A - \mu C^A + \lambda D^A,$$

where $f_i{}^h$ is a tensor field of type $(1,1)$, u_i, v_i, w_i 1-forms and λ, μ, ν functions in M^{2n+1}, u^h, v^h, and w^h being vector fields associated with u_i, v_i and w_i respectively.

Applying the operator F to both sides of (1.4)~(1.7), using (1.1) and those equations and comparing tangential parts and normal parts of both sides, we find

(1.8) $$f_i{}^t f_t{}^h = -\delta_i^h + u_i u^h + v_i v^h + w_i w^h,$$

$$(1.9)\qquad \begin{cases} f_t{}^h u^t = \nu v^h - \mu w^h, \\ f_t{}^h v^t = -\nu u^h + \lambda w^h, \\ f_t{}^h w^t = \mu u^h - \lambda v^h, \end{cases}$$

$$(1.10)\qquad \begin{cases} u_t u^t = 1 - \mu^2 - \nu^2, & u_t v^t = \lambda\mu, \\ v_t v^t = 1 - \nu^2 - \lambda^2, & v_t w^t = \mu\nu, \\ w_t w^t = 1 - \lambda^2 - \mu^2, & u_t w^t = \lambda\nu. \end{cases}$$

Also, from (1.1), (1.3) and (1.4), we find

$$(1.11)\qquad g_{ts} f_j{}^t f_i{}^s = g_{ji} - u_j u_i - v_j v_i - w_j w_i.$$

If we put $f_{ji} = f_j{}^t g_{ti}$, then we easily see that $f_{ji} = -f_{ij}$.

Thus (1.8)$\sim$(1.11) show that the aggregate $(f_i{}^h, g_{ji}, u_i, v_i, w_i, \lambda, \mu, \nu)$ defines the so-called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure on M^{2n+1} ([3], [6]).

An $(f, g, u, v, w, \lambda, \mu, \nu)$ -structure is said to be *antinormal* if the tensor field $S_{ji}{}^h$ of type (1, 2) defined by

$$(1.12)\qquad S_{ji}{}^h = [f, f]_{ji}{}^h + (\partial_j u_i - \partial_i u_j) u^h + (\partial_j v_i - \partial_i v_j) v^h + (\partial_j w_i - \partial_i w_j) w^h$$

satisfies

$$(1.13)\qquad S_{ji}{}^h = 2\{u_j(\partial_i u^h) - u_i(\partial_j u^h) + v_j(\partial_i v^h) - v_i(\partial_j v^h) + w_j(\partial_i w^h) - w_i(\partial_j w^h\},$$

where $[f, f]_{ji}{}^h$ is the Nijenhuis tensor formed with $f_i{}^h$, that is,

$$[f, f]_{ji}{}^h = f_j{}^t \partial_t f_i{}^h - f_i{}^t \partial_t f_j{}^h - (\partial_j f_i{}^t - \partial_i f_j{}^t) f_t{}^h.$$

We find from (1.9)

$$(1.14)\qquad f_t{}^h p^t = 0,$$

where we have put

$$(1.15)\qquad p^h = \lambda u^h + \mu v^h + \nu w^h.$$

From this and (1.10), we have

$$(1.16)\qquad u_t p^t = \lambda, \quad v_t p^t = \mu, \quad w_t p^t = \nu, \quad p_t p^t = \lambda^2 + \mu^2 + \nu^2.$$

We now suppose that the aggregate $(f_i{}^h, g_{ji}, p^h)$ defines an almost contact metric structure. Then we get from the last equation of (1.16)

$$(1.17)\qquad \lambda^2 + \mu^2 + \nu^2 = 1$$

because of $p_t p^t = 1$. Conversely if the function λ, μ and ν satisfy (1.17), then (1.10) reduces to

$$(1.18)\qquad \begin{aligned} &u_t u^t = \lambda^2, \quad u_t v^t = \lambda\mu, \quad u_t w^t = \lambda\nu, \\ &v_t v^t = \mu^2, \quad v_t w^t = \mu\nu, \quad w_t w^t = \nu^2. \end{aligned}$$

Hence, it follows that

$$(1.19)\qquad u_i = \lambda p_i, \quad v_i = \mu p_i, \quad w_i = \nu p_i$$

with the help of (1.16) and (1.18), where $p_i = g_{ti} p^t$. Substituting (1.19) into (1.8) gives $f_i{}^t f_t{}^h = -\delta_i^h + p_i p^h$ because of (1.17). Also substituting (1.19) into (1.11) and using (1.17), we find

$$g_{ts}f_j{}^tf_i{}^s=g_{ji}-p_jp_i.$$

Thus we see that the aggregate $(f_i{}^h, g_{ji}, p^h)$ defines an almost contact metric structure. Concluding the developed above, we have

THEOREM 1.1. ([6]) *Let M^{2n+1} be a differentiable manifold with an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure. In order for the aggregate (f, g, p), p being given by* (1.15), *to define an almost contact metric structure, it is necessary and sufficient that $\lambda^2+\mu^2+\nu^2=1$.*

In the sequel we suppose that the condition $\lambda^2+\mu^2+\nu^2=1$ is satisfied on M^{2n+1}. Suppose that the aggregate (f, g, p) defines an almost contact metric structure and the induced structure is antinormal. Then we have (1.19) and consequently (1.13) reduces to

$$[f,f]_{ji}{}^h+(\nabla_jp_i-\nabla_ip_j)p^h=2p_j(\nabla_ip^h)-2p_i(\nabla_jp^h) \tag{1.20}$$

with the help of (1.12) and (1.17). Thus we have

THEOREM 1.2. *Let M^{2n+1} be a differentiable manifold with an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. In order for this structure is antinormal, it is necessary and sufficient that* (1.20) *holds.*

2. Structure equations of submanifolds of codimension 3 of a Kaehlerian manifold

Suppose that aggregate (f, g, p) of $f_i{}^h, g_{ji}$ and $p^h=\lambda u^h+\mu v^h+\nu w^h$ defines an almost contact metric structure. Then we have (1.19) and consequently from (1.4)

$$F_C{}^AB_i{}^C=f_i{}^hB_h{}^A+p_iN^A \tag{2.1}$$

where $N^A=\lambda C^A+\mu D^A+\nu E^A$ is an intrinsically defined unit normal to M^{2n+1} because C^A, D^A and E^A are mutually orthogonal unit normals to M^{2n+1} and $\lambda^2+\mu^2+\nu^2=1$.

When a submanifold of an almost Hermitian manifold satisfies equation of the form (2.1), N^A being a unit normal to the submanifold, we say that the submanifold is *semi-invariant* with respect to N^A [1], [5]. We call N^A the *distinguished normal* to the semi-invariance. We take N^A as C^A. Then we have $\lambda=1$, $\mu=0$, $\nu=0$ and consequently $u^h=p^h$, $v^h=w^h=0$ because of (1.10) and (1.15). Thus (1.4)~(1.7) becomes respectively

$$F_C{}^AB_i{}^C=f_i{}^hB_h{}^A+P_iC^A, \tag{2.2}$$

$$F_B{}^AC^B=-p^hB_h{}^A, \tag{2.3}$$

$$F_B{}^AD^B=-E^A, \tag{2.4}$$

$$F_B{}^AE^B=D^A. \tag{2.5}$$

Now denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{ji}, we have equations of Gauss for M^{2n+1} of M^{2n+4}

(2.6) $$\nabla_j B_i{}^A = h_{ji} C^A + k_{ji} D^A + l_{ji} E^A,$$

where h_{ji}, k_{ji} and l_{ji} are the second fundamental tensors with respect to C^A, D^A and E^A respectively.

The equations of Weingarten are given by

(2.7) $$\nabla_j C^A = -h_j{}^h B_h{}^A + l_j D^A + m_j E^A,$$

(2.8) $$\nabla_j D^A = -k_j{}^h B_h{}^A - l_j C^A + n_j E^A,$$

(2.9) $$\nabla_j E^A = -l_j{}^h B_h{}^A - m_j C^A - n_j D^A.$$

where $h_j{}^h = h_{jt} g^{th}, k_j{}^h = k_{jt} g^{th}, l_j{}^h = l_{jt} g^{th}$, $(g^{ji}) = (g_{ji})^{-1}$, l_j, m_j and n_j being the third fundamental tensors. In the sequel we denote the normal components of $\nabla_j C^A$ by $\nabla_j{}^\perp C^A$. The normal vector field C^A is said to be *parallel* in the normal bundle if we have $\nabla_j{}^\perp C^A = 0$, i.e., l_j and m_j vanish identically.

We now assume that the ambient manifold M^{2n+4} is Kaehlerian. Differentiating (2.2) covariantly along M^{2n+1} and using (2.6) and (2.7), we easily find [6]

(2.10) $$\nabla_j f_i{}^h = -h_{ji} p^h + h_j{}^h p_i,$$

(2.11) $$\nabla_j p_i = -h_{jt} f_i{}^t,$$

(2.12) $$k_{ji} = -l_{jt} f_i{}^t - m_j p_i,$$

(2.13) $$l_{ji} = -k_{jt} f_i{}^t + l_j p_i,$$

from which

(2.14) $$k_{jt} p^t = -m_j,$$

(2.15) $$l_{jt} p^t = l_j,$$

(2.16) $$k = -m_t p^t,$$

(2.17) $$l = l_t p^t,$$

where we have put $k = g^{ji} k_{ji}, l = g^{ji} l_{ji}$.

Transvecting (2.13) with $f_k{}^j$ and making use of (2.12), we obtain

$$-k_{ih} - m_i p_h = k_{st} f_i{}^t f_h{}^s + (f_h{}^t l_t) p_i,$$

from which, taking the skew-symmetric part with respect to i and h,

$$m_h p_i - m_i p_h = p_i (l_t f_h{}^t) - p_h (l_t f_i{}^t),$$

or, transvecting with p^h and using (2.16)

(2.18) $$l_t f_i{}^t = k p_i + m_i.$$

If we transvect (2.18) with $f_k{}^i$ and l^i and take account of (2.17), we have respectively

(2.19) $$m_t f_h{}^t = l p_h - l_h,$$

(2.20) $$kl + m_t l^t = 0.$$

Transvecting (2.12) with $l_h{}^i$ and substituting (2.13), we find

$$k_{jt}l_h{}^t = -(l_{js}f_t{}^s + m_j p_t)(k_{hr}f^{tr} + l_h p^t),$$

or, using (2.14) and (2.15)

$$k_{jt}l_i{}^t + k_{it}l_j{}^t = -(l_j m_i + l_i m_j). \tag{2.21}$$

If we transvect (2.13) with $l_k{}^i$ and substitute (2.12), we have

$$l_{jt}l_i{}^t - k_{jt}k_i{}^t = l_j l_i - m_j m_i \tag{2.22}$$

with the help of (2.14) and (2.15).

Now suppose that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure is antinormal, that is,

$$f_j{}^t\nabla_t f_i{}^h - f_i{}^t\nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h + (\nabla_j p_i - \nabla_i p_j) p^h = 2p_j(\nabla_i p^h) - 2p_i(\nabla_j p^h)$$

by virtue of (1.20). Substituting (2.10) and (2.11) into this, we find

$$(f_j{}^t h_t{}^k + h_j{}^t f_t{}^k) p_i - (f_i{}^t h_t{}^k + h_i{}^t f_t{}^k) p_j = 0$$

and hence

$$f_j{}^t h_t{}^k + h_j{}^t f_t{}^k = p_j q^k, \qquad f_j{}^t h_t{}^k p_k = 0,$$

for a certain vector field q^k. From these equations we see that $q^k=0$, and consequently

$$h_{jt} f_i{}^t = h_{it} f_j{}^t. \tag{2.23}$$

Thus we have

THEOREM 2.1. *Suppose that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on a submanifold M^{2n+1} of codimension 3 of a Kaehlerian manifold M^{2n+4} satisfies $\lambda^2+\mu^2+\nu^2=1$. Then in order for this structure to be antinormal, it is necessary and sufficient that the second fundamental tensor h with respect to the distinguished normal and f anticommute.*

The Gauss equations of M^{2n+1} for a Kaehlerian manifold M^{2n+4} are given by

$$K_{kji}{}^h = K_{DCB}{}^A B_k{}^D B_j{}^C B_i{}^B B^h{}_A + h_k{}^h h_{ji} - h_j{}^h h_{ki} + k_k{}^h k_{ji} - k_j{}^h k_{ki} + l_k{}^h l_{ji} - l_j{}^h l_{ki}, \tag{2.24}$$

where $B^h{}_A = g_{AC}g^{jh}B_j{}^C$, $K_{kji}{}^h$ and $K_{DCB}{}^A$ being the Riemann–Christoffel curvature tensors of M^{2n+1} and M^{2n+4} respectively.

We now suppose that the ambient manifold is a Kaehlerian manifold $M^{2n+4}(c)$ of constant holomorphic sectional curvature c, that is, it's curvature tensor has the form

$$K_{DCB}{}^A = \frac{c}{4}(\delta_D{}^A g_{CB} - \delta_C{}^A g_{DB} + F_D{}^A F_{CB} - F_C{}^A F_{DB} - 2F_{DC}F_B{}^A). \tag{2.25}$$

Substituting (2.25) into (2.24) and taking account of (1.3), (2.2) and (2.3), we have

(2.26) $$K_{kji}{}^{h}=\frac{c}{4}(\delta_k^h g_{ji}-\delta_j^h g_{ki}+f_k{}^h f_{ji}-f_j{}^h f_{ki}-2f_{kj}f_i{}^h)$$
$$+h_k{}^h h_{ji}-h_j{}^h h_{ki}+k_k{}^h k_{ji}-k_j{}^h k_{ki}+l_k{}^h l_{ji}-l_j{}^h l_{ki}.$$

In the same way by using (2.2)$\sim$(2.5), we can prove that equations of the Codazzi for $M^{2n+4}(c)$ are given by

(2.27) $$\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}-m_k l_{ji}+m_j l_{ki}$$
$$=\frac{c}{4}(p_k f_{ji}-p_j f_{ki}-2p_i f_{kj}),$$

(2.28) $$\nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki}-n_k l_{ji}+n_j l_{ki}=0,$$

(2.29) $$\nabla_k l_{ji}-\nabla_j l_{ki}+m_k h_{ji}-m_j h_{ki}+n_k k_{ji}-n_j k_{ki}=0,$$

and those of the Ricci by

(2.30) $$\nabla_k l_j-\nabla_j l_k+h_k{}^t k_{jt}-h_j{}^t k_{kt}+m_k n_j-m_j n_k=0,$$

(2.31) $$\nabla_k m_j-\nabla_j m_k+h_k{}^t l_{jt}-h_j{}^t l_{kt}+n_k l_j-n_j l_k=0,$$

(2.32) $$\nabla_k n_j-\nabla_j n_k+k_k{}^t l_{jt}-k_j{}^t l_{kt}+l_k m_j-l_j m_k=\frac{c}{2}f_{kj}.$$

3. Submanifolds of codimension 3 of $M^{2n+4}(c)$ satisfying $\lambda^2+\mu^2+\nu^2=1$.

In this section we assume that the $(f, g, u, v, w, \lambda, \mu, \nu,)$-structure induced on a submanifold M^{2n+1} of codimension 3 of a Kaehlerian manifold $M^{2n+4}(c)$ of constant holomorphic sectional curvature c satisfies $\lambda^2+\mu^2+\nu^2=1$ and consequently the aggregate (f, g, p) defines an almost contact metric structure.

We now suppose that the sumanifold M^{2n+1} is umbilical with respect to the distinguished normal, that is, choosing C^A as the distinguished normal,

(3.1) $$h_{ji}=\tau g_{ji},\quad k=0,\quad l=0$$

for some function τ. Then (2.16), (2.17) and (2.20) imply that

(3.2) $$l_t p^t=m_t p^t=l_t m^t=0$$

and (2.11) becomes $\nabla_j p_i=\tau f_{ji}$, which shows that

$$\nabla_k\nabla_j p_i=(\nabla_k\tau)f_{ji}+\tau^2(g_{ki}p_j-g_{jk}p_i)$$

with the help of (2.10) and the first relation of (3.1), from which, using the Ricci identity,

$$-K_{kji}{}^{h}p_h=(\nabla_k\tau)f_{ji}-(\nabla_j\tau)f_{ki}+\tau^2(g_{ki}p_j-g_{ji}p_k),$$

or, taking account of the first Bianchi identity,

(3.3) $$(\nabla_k\tau)f_{ji}+(\nabla_j\tau)f_{ik}+(\nabla_i\tau)f_{kj}=0.$$

From this we can easily prove that τ is a constant. Thus (2.27) reduces to

(3.4) $$l_k k_{ji}-l_j k_{ki}+m_k l_{ji}-m_j l_{ki}=-\frac{c}{4}(p_k f_{ji}-p_j f_{ki}-2p_i f_{kj})$$

because of (3.1). Transvecting (3.4) with p^k and using (2.14), (2.15) and (3.2), we get

$$(3.5)\qquad m_j l_i - m_i l_j = \frac{c}{4} f_{ji}.$$

If we transvect (3.5) with f^{ji} and take account of (2.18), (2.19) and (3.1), then we get

$$(3.6)\qquad m_t m^t = \frac{c}{4} n.$$

Also, transvecting (3.5) with m^i and using (3.2) and (2.19) with $l=0$, we find

$$\left(m_t m^t - \frac{c}{4}\right) l_j = 0,$$

or substituting (3.6) into this, $cl_j=0$. Thus we have $c=0$ because of (3.5).

From (2.10), (2.11) and (3.1), we have

$$(3.7)\qquad \nabla_j f_i^{\ h} = \tau(-g_{ji} p^h + \delta_j^h p_i),\quad \nabla_j p_i = \tau f_{ji}.$$

Hence, it follows that the aggregate (f, g, p) defines a Sasakian structure if $\tau \neq 0$. We may consider $\tau=1$ because τ is a constant.

On the other hand, we see from (2.2) and (2.3) that the direct sum of the tangent space of M^{2n+1} and C^A is invariant. Then the ambient space being Euclidean, M^{2n+1} is an intersection of a complex cone with center at origin and with generator C^A and a $(2n+3)$-dimensional sphere (See [6]). Thus we have

THEOREM 3.1. *Let M^{2n+1} be a umbilical submanifold with respect to the distinguished normal C^A of a Kaehlerian manifold $M^{2n+4}(c)$ of constant holomorphic sectional curvature c satisfying $\lambda^2+\mu^2+\nu^2=1$. Then M^{2n+1} is an intersection of a complex cone with generator C^A and a sphere.*

We next prove the following

THEOREM 3.2. *Let M^{2n+1} be a submanifold of codimension 3 of a Kaehlerian manifold $M^{2n+1}(c)$ of constant holomorphic sectional curvature c with antinormal $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If the distinguished normal C^A is parallel in the normal bundle and the third fundamental tensor n_j satisfies*

$$(3.8)\qquad \nabla_j n_i - \nabla_i n_j = 2\alpha f_{ji}$$

for a certain function α, then M^{2n+1} is a hypersurface of $M^{2n+2}(c)$.

Proof. Since $\nabla_j^{\perp} C^A=0$, that is, l_j and m_j vanish identically, we have from (2.32)

$$\nabla_k n_j - \nabla_j n_k + 2k_k^{\ t} l_{jt} = \frac{c}{2} f_{kj}$$

because of (2.21). Thus (3.8) reduces to

$$k_k{}^t l_{jt}=\left(\frac{c}{4}-\alpha\right)\cdot f_{kj}. \tag{3.9}$$

Transvecting (3.9) with $f_i{}^k$ and taking account of (2.13) with $l_j=0$, we find

$$l_{jt}l_i{}^t=\left(\alpha-\frac{c}{4}\right)\cdot(g_{ji}-p_jp_i). \tag{3.10}$$

Therefore, it follows that

$$k_{jt}k_i{}^t=\left(\alpha-\frac{c}{4}\right)\cdot(g_{ji}-p_jp_i) \tag{3.11}$$

because of (2.22) with $l_j=m_j=0$.

Since $l_j=m_j=0$, (2.28), (2.29) and (2.31) reduces respectively to

$$\nabla_k k_{ji}-\nabla_j k_{ki}=n_k l_{ji}-n_j l_{ki}, \tag{3.12}$$

$$\nabla_k l_{ji}-\nabla_j l_{ki}=-n_k k_{ji}+n_j k_{ki}, \tag{3.13}$$

$$h_k{}^t l_{jt}-h_j{}^t l_{kt}=0. \tag{3.14}$$

Transvecting (3.14) with $l_i{}^k$ and making use of (3.10), we find

$$\left(\alpha-\frac{c}{4}\right)\cdot(h_{ji}-p_ih_{jt}p^t)-h_{st}l_j{}^s l_i{}^t=0,$$

from which, taking the skew-symmetric part,

$$\left(\alpha-\frac{c}{4}\right)\cdot(h_{jt}p^t-\beta p_j)=0, \tag{3.15}$$

where we have put

$$\beta=h_{st}p^sp^t. \tag{3.16}$$

As in the proof of Theorem 3.1, we can easily from (3.8) see that α is a constant by using the Ricci and Bianchi identities.

Differentiating (3.10) covariantly and using the fact that $\alpha-\frac{c}{4}$ is a constant, we obtain

$$l_i{}^t(\nabla_k l_{jt})+l_j{}^t(\nabla_k l_{it})=\left(\frac{c}{4}-\alpha\right)\cdot\{(\nabla_k p_j)p_i+(\nabla_k p_i)p_j\}, \tag{3.17}$$

from which, taking the skew-symmetric part with respect to k and j and substituting (3.13),

$$l_i{}^t(n_jk_{kt}-n_kk_{jt})+l_j{}^t(\nabla_i l_{kt}-n_kk_{it}+n_ik_{kt})-l_k{}^t(\nabla_i l_{jt}-n_jk_{it}+n_ik_{jt})$$
$$=\left(\frac{c}{4}-\alpha\right)\cdot\{(\nabla_k p_j-\nabla_j p_k)p_i+(\nabla_k p_i)p_j-(\nabla_j p_i)p_k\},$$

or using (2.21) with $l_j=m_j=0$,

$$l_j{}^t(\nabla_i l_{kt})-l_k{}^t(\nabla_i l_{jt})+2n_il_j{}^tk_{kt}$$
$$=\left(\frac{c}{4}-\alpha\right)\cdot\{(\nabla_k p_j-\nabla_j p_k)p_i+(\nabla_k p_i)p_j-(\nabla_j p_i)p_k\}.$$

Interchanging the indices k and i, we get

$$l_j^t(\nabla_k l_{it}) - l_i^t(\nabla_k l_{jt}) + 2n_k l_j^t k_{it} = \left(\frac{c}{4}-\alpha\right)\cdot\{(\nabla_i p_j - \nabla_j p_i)p_k + (\nabla_i p_k)p_j - (\nabla_j p_k)p_i\}. \tag{3.18}$$

Adding (3.17) to (3.18), we find

$$2l_j^t\nabla_k l_{it} + 2n_k l_{jt} k_i^t = \left(\frac{c}{4}-\alpha\right)\cdot\{(\nabla_k p_j - \nabla_j p_k)p_i + (\nabla_k p_i + \nabla_i p_k)p_j + (\nabla_i p_j - \nabla_j p_i)p_k\},$$

from which, transvecting p^j and taking account of (2.11), (2.15) with $l_j=0$ and (3.15),

$$\left(\frac{c}{4}-\alpha\right)\cdot(h_{kt}f_i^t + h_{it}f_k^t) = 0. \tag{3.19}$$

Since the induced structure is antinormal, by transvecting f_j^k and taking account of (2.23) and (3.15), we find

$$\left(\frac{c}{4}-\alpha\right)\cdot(h_{ji} - \beta p_j p_i) = 0. \tag{3.20}$$

If $\frac{c}{4}-\alpha\neq 0$, then (2.11) becomes $\nabla_j p_i = 0$ because of $h_{ji} = \beta p_j p_i$. Thus (2.27) reduces to

$$(\nabla_k\beta)p_j p_i - (\nabla_j\beta)p_k p_i = \frac{c}{4}\left(p_k f_{ji} - p_j f_{ki} - 2p_i f_{kj}\right) \tag{3.21}$$

because of $l_j = m_j = 0$.

Transvecting (3.21) with $p^j p^i$, we obtain $\nabla_k\beta = (p^t\nabla_t\beta)p_k$. Hence (3.21) implies that c is zero. Consequently the ambient manifold is Euclidean. According to Lemma 5.4 of [6], α must be zero. It contradicts the fact that $\frac{c}{4}-\alpha\neq 0$. Thus we have $\alpha=\frac{c}{4}$. Thereby (2.26)$\sim$(2.32) become the structure equations for a hypersurface of $M^{2n+2}(c)$. Thus we complete the proof of the theorem.

4. Antinormal submanifolds of codimension 3 of $M^{2n+4}(c)$ satisfying $\lambda^2+\mu^2+\nu^2=1$.

In this section we assume that the induced $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on a submanifold M^{2n+1} of codimension 3 of a Kaehlerian manifold $M^{2n+4}(c)$ of constant holomorphic sectional curvature $c\geq 0$ satisfies $\lambda^2+\mu^2+\nu^2=1$ and is antinormal. Then we have (2.23).

Transvecting (2.23) with p^i and taking account of (1.14), we get

$$(h_{it}p^i)f_j^t = 0,$$

from which,

(4.1) $$h_{it}p^t=\beta p_i$$

because of (3.16).

Differentiating (4.1) covariantly and substituting (2.11), we find

$$(\nabla_j h_{it})p^t-h_i{}^t h_{js}f_t{}^s=(\nabla_j\beta)p_i-\beta h_{jt}f_i{}^t,$$

from which, taking the skew-symmetric part and using (2.23) and (2.27)

$$\left\{l_jk_{it}-l_ik_{jt}+m_jl_{it}-m_il_{jt}+\frac{c}{4}(p_jf_{it}-p_if_{jt}-2p_tf_{ji})\right\}p^t$$
$$=2h_i{}^th_{js}f_t{}^s+(\nabla_j\beta)p_i-(\nabla_i\beta)p_j,$$

or, using (2.14), (2.15) and (2.23)

(4.2) $$h_i{}^th_{st}f_j{}^s+\frac{c}{4}f_{ji}=\frac{1}{2}\{(\nabla_i\beta)p_j-(\nabla_j\beta)p_i\}+l_im_j-l_jm_i.$$

If we transvect (4.2) with p^j, then we have

(4.3) $$\frac{1}{2}\nabla_i\beta=\frac{1}{2}(p^t\nabla_t\beta)p_i+lm_i+kl_i$$

because of (2.16) and (2.17). Thus (4.2) gives

(4.4) $$h_i{}^th_{st}f_j{}^s+\frac{c}{4}f_{ji}=l(m_ip_j-m_jp_i)+k(l_ip_j-l_jp_i)+l_im_j-l_jm_i.$$

Transvecting (4.4) with $f_k{}^j$ and using (4.1), we find

$$-h_i{}^th_{kt}+\beta^2p_ip_k+\frac{c}{4}(-g_{ik}+p_ip_k)$$
$$=(l_i-lp_i)f_k{}^tm_t-(m_i-kp_i)f_k{}^tl_t,$$

from which, substituting (2.18) and (2.19),

(4.5) $$h_i{}^th_{kt}-\beta^2p_ip_k+\frac{c}{4}(g_{ik}-p_ip_k)$$
$$=l_il_k+m_im_k-l(l_ip_k+l_kp_i)+k(m_ip_k+m_kp_i)+(l^2+k^2)p_ip_k.$$

On the other hand, transvecting (2.23) with f^{ji} and making use of (4.1), we have

(4.6) $$h=\beta,$$

where we have put $g^{ji}h_{ji}=h$.

Using this fact, (4.5) reduces to

(4.7) $$h_{jt}h_i{}^t+\frac{c}{4}g_{ji}=\left(h^2+k^2+l^2+\frac{c}{4}\right)p_jp_i+l_jl_i+m_jm_i$$
$$+k(m_jp_i+m_ip_j)-l(l_jp_i+l_ip_j),$$

which implies

(4.8) $$h_{ji}h^{ji}=h^2-k^2-l^2+l_tl^t+m_tm^t-\frac{nc}{2}$$

with the help of (2.16) and (2.17). Since the left hand side of (4.8) becomes $\|h_{ji}-hp_jp_i\|^2$ because of (4.1) and (4.6), (4.8) can be written as

$$\|h_{ji}-hp_jp_i\|^2=l_tl^t+m_tm^t-\left(k^2+l^2+\frac{nc}{2}\right). \tag{4.9}$$

For an eigenvalue ρ of $h_j{}^i$ corresponding to the eigenvector orthogonal to p^i, l^i and m^i, we have from (4.7) that $\rho^2+\frac{c}{4}=0$ if $n\geqq 2$. Thus it follows that $c\leqq 0$ because the eigenvalue is real and hence $c=0$.

We now suppose that $\nabla_j{}^{\perp}C^A=0$ and M^{2n+1} is minimal. Then we have from (4.8) with $c=0$ that $h_{ji}=0$. Therefore (2.26)$\sim$(2.32) mean that M^{2n+1} is a submanifold of codimension 2 in a Euclidean space E^{2n+3} because of $c=0$.

Hence we have

PROPOSITION 4.1. *Let M^{2n+1} $(n\geqq 2)$ be a minimal submanifold of codimension 3 of a Kaehlerian manifold $M^{2n+4}(c)$ of constant holomorphic sectional curvature $c\geqq 0$ such that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on M^{2n+1} defines an almost contact metric structure (f, g, p), p being given by* (1.15) *and is anti-normal. If the distinguished normal C^A is parallel in the normal bundle, then M^{2n+1} is a submanifold of a Euclidean space E^{2n+3}.*

Denoting by $K_{ji}=K_{tji}{}^t$ and $K=g^{ji}K_{ji}$ the Ricci tensor and the scalar curvature of M^{2n+1} respectively, we then have from (2.26)

$$K_{ji}=\frac{c}{4}\{(2n+3)g_{ji}-3p_jp_i\}+hh_{ji}+kk_{ji}+ll_{ji}$$
$$-h_{jt}h_i{}^t-k_{jt}k_i{}^t-l_{jt}l_i{}^t,$$

from which

$$K=n(n+2)\cdot c+h^2+k^2+l^2-h_{ji}h^{ji}-k_{ji}k^{ji}-l_{ji}l^{ji},$$

or, substituting (4.8) and taking account of (2.22)

$$K=\frac{n(2n+5)}{2}c-2(l_tl^t-l^2)-2(k_{ji}k^{ji}-k^2),$$

which means

$$K=\frac{n(2n+5)}{2}c-\|l_jp_i-l_ip_j\|^2-2\|k_{ji}-kp_jp_i\|^2 \tag{4.10}$$

with the help of (2.14)$\sim$(2.17). Thus if $K\geqq\frac{n(2n+5)}{2}c$ holds, we have $l_jp_i-l_ip_j=0$, $k_{ji}=kp_jp_i$. Hence (2.12) and (2.13) imply that $l_{ji}=lp_jp_i$, $m_jp_i=m_ip_j$. It follows from (4.9) that $\|h_{ji}-hp_jp_i\|^2+\frac{n}{2}c=0$ and consequently $h_{ji}=hp_jp_i$ and $c=0$ because of $c\geqq 0$, Thus (2.10) and (2.11) becomes $\nabla_jf_i{}^h=0$, $\nabla_jp_i=0$. And (2.26) reduces to $K_{kji}{}^h=0$.

Therefore we have

PROPOSITION 4.2. *Let M^{2n+1} be a submanifold of codimension 3 of a Kaehlerian manifold $M^{2n+4}(c)$ of constant holomorphic sectional curvature $c \geqq 0$ such that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on M^{2n+1} is antinormal and satisfies $\lambda^2+\mu^2+\nu^2=1$. If the scalar curvature K of M^{2n+1} satisfies $K \geqq \frac{n(2n+5)}{2}c$ at every point, then M^{2n+1} is a locally Euclidean space with the second fundamental tensors of the forms*

$$h_{ji}=hp_jp_i, \qquad k_{ji}=kp_jp_i, \qquad l_{ji}=lp_jp_i$$

and admits a cosymplectic structure.

We now prove the following

THEOREM 4.3. *Let M^{2n+1} be a complete submanifold of codimension 3 of a Euclidean space E^{2n+4} with antinormal $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If the distinguished normal C^A is parallel in the normal bundle and the third fundamental tensor of M^{2n+1} satisfies*

$$(4.11) \qquad \nabla_j n_i - \nabla_i n_j = 2\alpha f_{ji},$$

then M^{2n+1} is a plane or a ruled surface which is generated by parallel displacements of a plane E^{2n} along a plane curve orthogonal to E^{2n}.

Proof. Since $\nabla_j{}^{\perp} C^A=0$, that is, $l_j=m_j=0$, we have from (4.9) with $c=0$

$$(4.12) \qquad h_{ji}=hp_jp_i.$$

From (4.11) we can prove that $\alpha=0$ and hence

$$(4.13) \qquad k_{ji}=0, \quad l_{ji}=0.$$

(See Lemma 5.4 and Theorem 5.5 of [6]). Thus (2.7)~(2.9) reduce to respectively

$$(4.14) \qquad \nabla_j C^A=-hp_j(p^h B_h{}^A), \quad \nabla_j D^A=n_j E^A, \quad \nabla_j E^A=-n_j D^A$$

because of $l_j=m_j=0$. Also (2.11) and (4.12) imply that

$$(4.15) \qquad \nabla_j p^h=0.$$

Let M' be a real hypersurface of M^{2n+1} which is defined by the Pfaffian form $\omega=p_i dx^i$ and be covered by a system of coordinate neighborhoods $\{U' ; \xi^a\}$, where the indices a, b, c run over the range $1', 2', \dots, 2n'$.

Let $i' : M' \to M^{2n+1}$ be an isometric immersion represented by $y^h=y^h(\xi^a)$. Putting $B_a{}^h=\partial_a y^h$, $(\partial^a=\partial/\partial\xi^a)$, then $B_a{}^h$ are $2n$ linearly independent vectors of M^{2n+1} tangent to M'. By definition, p^h is a unit normal to M'. Now we put

$$(4.16) \qquad B_a{}^A=B_a{}^h B_h{}^A, \quad P^A=p^h B_h{}^A.$$

Then P^A is a unit normal vector field orthogonal to C^A, D^A and E^A. In this case, we can easily see that M' is a totally geodesic submanifold of E^{2n+4}

because of (4.12), (4.13) and (4.15). Consequently M' is a plane E^{2n} parallel along p^h because the ambient space is Euclidean.

If we take account of (4.12) and (4.13), then (2.6) becomes $\nabla_j B_h{}^A = hp_jp_hC^A$, or by transvecting p^h

$$\nabla_j P^A = hp_jC^A \tag{4.17}$$

with the help of (4.15).

From (4.14) and (4.17), we have

$$p^j\nabla_j C^A = -hP^A, \qquad p^j\nabla_j P^A = hC^A,$$

which shows a plane curve with curvature h on a complex two dimensional plane C^2 spaned by $\{P^A, C^A, D^A, E^A\}$. Then the orthogonal complementary space of C^2 is a plane E^{2n}. Hence M^{2n+1} is a ruled surface which is generated by parallel displacements of E^{2n} along a curve on C^2 if $h\neq 0$. If $h=0$, then M^{2n+1} is a plane in E^{2n+4} because of (4.12) and (4.13). This completes the proof the theorem.

Replacing the conidtion (4.11) in Theorem 4.3 by $K\geqq 0$, we can see that $k_{ji}=0$, $l_{ji}=0$. In fact, since $\nabla_j{}^{\perp}C^A=0$, (4.9) with $c=0$ implies that $h_{ji}=hp_jp_i$. Consequently (4.10) with $c=0$ becomes $K=-2k_{ji}k^{ji}$ with the help of $l_j=0$ and $k=0$. It follows that $k_{ji}=0$ becauuse of $K\geqq 0$ and hence $l_{ji}=0$ by virtue of (2.22).

According to Theorem 4.3, we have

COROLLARY 4.4. *Let M^{2n+1} be a complete submanifold of codimension 3 of a Euclidean space E^{2n+4} with antinormal $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If the distinguished normal C^A is parallel in the normal bundle and the scalar curvature of M^{2n+1} is nonnegative at every point, we have the same conclusions of Theorem 4.3.*

5. Submanifolds of codimension 3 of E^{2n+4} whose normal connection is flat.

In this section we assume that the connection induced in the normal bundle of M^{2n+1} in a Euclidean space E^{2n+4} is flat. Then we have

$$h_j{}^tk_{ti}-h_i{}^tk_{tj}=0,\quad h_j{}^tl_{ti}-h_i{}^tl_{tj}=0,\quad k_j{}^tl_{ti}-k_i{}^tl_{jt}=0. \tag{5.1}$$

Transvecting (2.12) with k^{ji} and using the third relation of (5.1), we find $k_{ji}k^{ji}=m_tm^t$, from which, using (2.14), $\|k_{ji}+m_jp_i\|^2=0$ and consequently

$$k_{ji}=-m_jp_i. \tag{5.2}$$

If we take the skew-symmetric part of this, then we have $m_jp_i=m_ip_j$, or by using (2.16), $m_j=-kp_j$. Thus (5.2) becomes

$$k_{ji}=kp_jp_i. \tag{5.3}$$

In the same way we have from (2.13), (2.15) and (2.17) that

(5.4) $$l_{ji}=lp_jp_i, \quad l_j=lp_j.$$

Thus (4.9) with $c=0$ implies

(5.5) $$h_{ji}=hp_jp_i$$

with the help of the fact that $l_tl^t=l^2$ and $m_tm^t=k^2$.

Conversely, if (5.3)~(5.5) are satisfied, then we easily see that (2.23) and (5.1) are valid. Therefore we have

PROPOSITION 5.1. *Suppose that the* $(f, g, u, v, w, \lambda, \mu, \nu)$*-structure induced on a submanifold* M^{2n+1} *of codimension* 3 *of a Euclidean space* E^{2n+4} *satisfies* $\lambda^2+\mu^2+\nu^2=1$ *and consequently* (f, g, p) *defines an almost contact metric structure. Then in order for these structures to be antinormal and the connection induced in the normal bundle of* M^{2n+1} *to be trivial, it is necessary and sufficient that the second fundamental tensors of* M^{2n+1} *have the form*

(5.6) $$h_{ji}=hp_jp_i, \quad k_{ji}=kp_jp_i, \quad l_{ji}=lp_jp_i.$$

On the other hand, the mean curvature vector H of M^{2n+1} is given by

$$H=\frac{1}{2n+1}(hC+kD+lE).$$

If we now take the distinguished normal as a direction of the mean curvature vector if $H\neq 0$, that is, we choose the normals $H/\|H\|$, $'D$ and $'E$ such that $H=\|H\|C$, then we have

$$\begin{pmatrix} 'C \\ 'D \\ 'E \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}\cdot\begin{pmatrix} C \\ D \\ E \end{pmatrix}$$

for some constant θ, where $'C=H/\|H\|$. This means

(5.7) $$'C=C, \quad 'D=\cos\theta\cdot D+\sin\theta\cdot E, \quad 'E=-\sin\theta\cdot D+\cos\theta\cdot E.$$

As to the transforms of $B_i{}^A$, $'C^A$, $'D^A$ and $'E^A$ by $F_B{}^A$, we have respectively the equations of the form

$$\begin{aligned} F_B{}^AB_i{}^B&=f_i{}^hB_h{}^A+'u_i'C^A+'v_i'D^A+'w_i'E^A, \\ F_B{}^A{}'C^B&=-'u^hB_h{}^A-'\nu'D^A+'\mu'E^A, \\ F_B{}^A{}'D^B&=-'v^hB_h{}^A+'\nu'C^A-'\lambda'E^A, \\ F_B{}^A{}'E^B&=-'w^hB_h{}^A-'\mu'C^A+'\lambda'E^A. \end{aligned}$$

If we apply the operator F to these equations and use (5.7), we obtain

(5.8) $$'\lambda=\lambda, \quad '\mu=\cos\theta\cdot\mu+\sin\theta\cdot\nu, \quad '\nu=-\sin\theta\cdot\mu+\cos\theta\cdot\nu,$$

which shows that $'\lambda=1$, $'\mu='\nu=0$ if $\lambda=1$, $\mu=\nu=0$, that is, although the normals C, D and E are rotated by the fixed angle θ, we may take H as distinguished normal.

Let $'h_{ji}$, $'k_{ji}$ and $'l_{ji}$ be the second fundamental tensors with respect to $'C, 'D$ and $'E$, and $'l_j, 'm_j$ and $'n_j$ the third fundamental tensors corresponding to l_j, m_j and n_j respectively.

By differentiating (5.7) covariantly and taking account of (2.7)~(2.9), we then have

$$(5.9)\qquad 'h_{ji}=h_{ji},\quad 'k_{ji}=\cos\theta\cdot k_{ji}+\sin\theta\cdot l_{ji}$$
$$'l_{ji}=-\sin\theta\cdot k_{ji}+\cos\theta\cdot l_{ji},$$

$$(5.10)\qquad 'l_j=\cos\theta\cdot l_j+\sin\theta\cdot m_j,\quad 'm_j=-\sin\theta\cdot l_j+\cos\theta\cdot m_j,\quad 'n_j=n_j$$

because θ is a constant.

Since the distinguished normal as a direction of the mean curvature vector, we have

$$(5.11)\qquad 'h=h,\quad 'k='l=0,$$

where we have put $'h='h_t{}^t$ and $'k='k_t{}^t$ and $'l='l_t{}^t$.

By using (5.9), we can easily verify that (2.23) and (5.1) are of intrinsic characters. Hence (5.6) implies

$$'h_{ji}=hp_jp_i,\quad 'k_{ji}='l_{ji}=0$$

because of (5.11). As in the proof of Theorem 4.3, M^{2n+1} is a ruled surface which is generated by parallel displacements of a plane E^{2n} along a plane curve orthogonal to E^{2n} if $H\neq 0$. Thus we have

THEOREM 5.2. *Let M^{2n+1} be a complete submanifold of codimension 3 of a Euclidean space E^{2n+4} with antinormal $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$ whose normal connection is flat. If we take the distinguished normal as a direction of the mean curvature vector H, then M^{2n+1} is a ruled surface which is generated by parallel displacements of a plane E^{2n} along a plane curve orthogonal to E^{2n} provided that $H\neq 0$. If $H=0$, then M^{2n+1} is a plane E^{2n+1}.*

Combining Proposition 4.2 and Theorem 5.2, we have

COROLLARY 5.3. *Let M^{2n+1} be a complete submanifold of codimension 3 of a Euclidean space E^{2n+4} with antinormal $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If we take the distinguished normal as a direction of the mean curvature vector and the scalar curvature of M^{2n+1} is nonnegative at every point, we have the same conclusions of Theorem 5.2.*

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Sung Kyun Kwan University, Kyungpook National University