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REMARKS ON REDUCING OPERATOR VALUED SPECTRUM

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1. Introduciton

In [3] and [4], D. W. Hadwin initiated the study of reducing operator valued spectrum, and made further progresses in his subsequent papers ([5] [6] [7]).

The aim of this article is to provide still other informati on which appear to be overlooked in the Hadwin's works above. Throughout, H denotes a separable infinite dimensional Hilbert space over the complex numbers, $\mathcal{B}(H)$ the set of all operators (bounded linear transformations) on H, and $\mathcal{K}(H)$ the ideal of compact operators in $\mathcal{B}(H)$. A closed linear manifold M of H will be called a subspace of H and denoted by $M \leq H$.

For more technical terminologies and notations, we shall follow [4], with little changes.

2. Reducing Operator Eigenvalue

The next definition extends the corresponding one in ([4] p. 332), by removing the irreducibility requirement for the operator A.

DEFINITION 1. If $T \in \mathcal{B}(H)$ and $A \in \mathcal{B}(K)$, where K is any nonzero separable Hilbert space. Then the reducing eigenspace Eig $(A; T)$ of A is defined as the set of all vectors $f \in H$ such that $p_n(T, T^*)f \to 0$ weakly in H, whenever $\{p_n(x, y)\}$ is a sequence of noncommutative polynomials such that $p_n(A, A^*) \to 0$ in the weak opeator topology in $\mathcal{B}(K)$.

The next lemma is an easy consequence of definition 1 and hence the proof is omitted.

LEMMA 1. *Let T, A be as in definition 1. If M is a reducing subspace of T and $T|M$ is unitarily equivalent to A, denoted by $T|M \cong A$, then* $M \subset Eig(A; T)$.

DEFINITION 2. Let K and L be nonzero separable Hilbert spaces. Let $A \in \mathcal{B}(K)$ and $B \in \mathcal{B}(L)$. Then A and B are called disjoint, denoted by

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$A \circ B$, if no suboperator of S is unitarily equivalent to any suboperator of T (definition 2.1 [4] p. 332).

The following proposition extends (i) and (ii) of proposition 2.3 ([4] p. 332).

PROPOSITION 1. *Let T, A be as in definition 1. Then $Eig(A;T)$ is a reducing subspace of T.*

Proof. It is routine to check that $Eig(A;T)$ is a linear submanifold of H invariant under T and T^*. We denote by M the norm closure of $Eig(A;T)$. It suffices to show that $M \subset Eig(A;T)$. Without loss of generality, we may assume that $Eig(A;T) \neq \{0\}$, so that $M \neq \{0\}$. First we will show that $T|M$ has no suboperater disjoint from A. Assume, to the contrary, that $T|M$ has a suboperator $B \in \mathcal{B}(N)$ such that $B \circ A$.

By (ii) of Lemma 2.2 ([4] p. 332), there is a sequence $\{p_n(x,y)\}$ of non-commutative polynomials such that $p_n(A, A^*) \to 0$ in the weak operator topology on $\mathcal{B}(K)$ and $P_n(B, B^*) \to I$ in the weak operator topology on $\mathcal{B}(N)$. Let us pick up $g \neq 0$ in N. Then

(1) $\quad (p_n(T, T^*)g, g) = (p_n(B, B^*)g, g) \to (g, g).$

Now $\{p_n(B, B^*)\}$ is a norm bounded sequence, say by $k(>0)$, from the uniform boundedness principle. Pick up a vector $f \in Eig(A;T)$ such that $\|f-g\| < \|g\|/(2k)$. Then

$$\begin{aligned} |(p_n(T, T^*)g, g)| &\leqq |(p_n(T, T^*)(g-f), g)| + |(p_n(T, T^*)f, g)| \\ &\leqq \|p_n(B, B^*)\| \ \|g-f\| \ \|g\| + |p_n(T, T^*)f, g)| \\ &\leqq k(\|g\|/2k)\|g\| + (1/4)\|g\|^2, \end{aligned}$$

(for sufficiently large n.)

$$= (3/4)\|g\|^2.$$

This contradicts to (1), showing that no suboperator of $T|M$ is disjoint from A. In other words,

(2) every suboperator B of $T|M$ has a suboperator D (of B) that is unitarily equivalent to a suboperator A_1 of A.

Let $\mathcal{F} = \{\sum_{\alpha \in J} \oplus D_\alpha;\ J \in \Lambda\}$ be the set of all operators of the form $\sum_{\alpha \in J} \oplus D_\alpha$, where each D_α is a suboperator of $T|M$ and D_α is unitarily equivalent to a suboperator A_α of A. Then $\mathcal{F}$ is nonempty, by (2). Here we understand that $D_\alpha \neq D_\beta$, whenever $\alpha \neq \beta, \alpha, \beta \in J$ and $D_\alpha \in \mathcal{B}(M_\alpha)$. We order by $\sum_{\alpha \in J_1} \oplus D_\alpha \leqq \sum_{\alpha \in J_2} \oplus D_\alpha$, if $J_1 \subset J_2$. Then $\leqq$ is a partial ordering for $\mathcal{F}$. We can apply Zorn's lemma to $\mathcal{F}$ and use (2) to conclude that $T|M = \sum_{\alpha \in I} \oplus D_\alpha$, for certain $I \in \Lambda$. It follows that, for some unitary operator

$U:\sum_{\alpha\in I}\oplus M_\alpha\to M$, we have

$$U^*(T|M)=\sum_{\alpha\in I}\oplus A_\alpha\in\mathcal{B}(\sum_{\alpha\in I}\oplus M_\alpha)$$

Let $f\in M$. We want to show that $f\in Eig(A;T)$. Assume that $p_n(A,A^*)\to 0$ in the weak operator topology in $\mathcal{B}(K)$. It suffices to show that

(3) $(p_n(T,T^*)f,g)\to 0$ as $n\to\infty$, for each $g\in M$.

Put $U^*f=\xi_1\oplus\xi_2\oplus\cdots\in\sum_{\alpha\in I}\oplus M_i$. Then

$$\begin{aligned}(p_n(\sum_{\alpha\in I}\oplus A_i,\sum_{\alpha\in I}\oplus A_i^*)\xi,\xi)&=((\sum_{i\in I}\oplus p_n(A_i,A_i^*))\xi,\xi)\\&=\sum_{i\in I}(p_n(A_i,A_i^*)\xi_i,\xi_i),\end{aligned}$$

and

$$\|p_n(A_i,\ A_i^*)\|_{M_i}\leqq\|p_n(A,A^*)\|_K,$$

while $\{\|p_n(A,A^*)\|_K\}$ is a bounded sequence.

Thus, to show (3), it only needs to verify that

$$(p_n(U(\sum_{i\in I}\oplus A_i)U^*,\ U(\sum_{i\in I}\oplus A_i^*)U^*)f,\ g)\to 0.$$

By a simple computation, this is equivalent to say that

$$\sum_{i\in I}(p_n(A,A^*)\xi_i,\eta_i)\to 0, \tag{4}$$

where $U^*g=\eta_1\oplus\eta_2\oplus\cdots\in\sum_{i\in I}\oplus M_i$.

By Dixmier [1] p. 34, the weak operator topology on $\mathcal{B}(K)$ coincides with the ultraweak operator topology on a norm bounded subset of $\mathcal{B}(K)$. Hence (4) holds, by the fact that $p_n(A,A^*)\to 0$ in the weak operator topology in $\mathcal{B}(K)$. It follows that (3) holds and $M\subset Eig(A;T)$, that is $M=Eig(A;T)$. *Q.E.D.*

A quick review of the proof of proposition 1 yields the fact:

if $Eig(A;\ T)\neq\{0\}$, then $T|Eig(A;T)$ has no suboperator disjoint from A, which is called that A covers $T|Eig(A;T)$, denoted by $A\}(T|Eig(A;T))$ (Ernest [2] p. 9 definitien 1.10).

By modifying the proof of lemma 2, we can say a little bit more as follows.

PROPOSITION 2. *Let T, A be as in definition 1. Let A_1 be a suboperator of A. Then*

(i) *$Eig(A_1;T)\subset Eig(A;\ T)$,*

(ii) *if $Eig(A_1;T)\neq\{0\}$, then $A_1\}(T|Eig(A;\ T)$.*

We say that two operators S and T are quasiequivalent, denoted by $S\approx T$, if $S\}T$ and $T\}S$ (Ernest [2] p. 9, definition 1.10).

PROPOSITION 3. *Let A, B and T be operators acting on nonzero separable Hilbert spaces K, L and H respectively. Then*

(i) $A \mathbin{\overset{|}{\circ}} B$ *if and only if* $Eig(A;T) \perp Eig(B;T)$.

If, in addition, A and B are factor operators, then

(ii) $A \approx B$ *if and only if* $Eig(A;T) = Eig(B;T)$, *when* $Eig(A;T) \neq \{0\}$.

Proof. (i) ($\Rightarrow$) In the proof of proposition 2.3 (iv) [4] p. 333, only the fact that $A \mathbin{\overset{|}{\circ}} B$ is used. Hence we can follow the proof there.

($\Leftarrow$) Assume that $Eig(A;T) \perp Eig(B;T)$. Suppose, contrarily that $A \not\mathbin{\overset{|}{\circ}} B$. Then there are suboperators C of A and D of B such that $C \cong D$.

Since $Eig(C;T) \leqq Eig(A;T)$, we have $Eig(D;T) \leqq Eig(B;T)$, by proposition 2(i) and $Eig(C;T) = Eig(D;T)$, we have a contradiction.

(ii) By corollary 1.4 (Ernest [2] p. 12), either $A \lesssim B$ or $B \lesssim A$. (Here, $A \lesssim B$ means that A is unitarily equivalent to a suboperator of B). If both $A \lesssim B$ and $B \lesssim A$, then $A \cong B$ by Theorem 1.3 ([2] p. 6, Ernest), in which case $Eig(A;T) = Eig(B;T)$. So assume that $B \lesssim A$ with $A = A_1 \oplus A_2$, $B \cong A_1$ and $A_2 \lnsim A$. (Recall that operators are always assumed to be acting on nonzero spaces.) Clearly, $Eig(B;T) = Eig(A_1;T) \subset Eig(A;T)$. Now we show the reverse inclusion, $Eig(A;T) \subset Eig(B;T)$. Put

$$N = Eig(A;T) \ominus Eig(A_1;T) \neq \{0\}.$$

We apply (ii) of proposition 2, we see that $A_1\} (T|Eig(A;T))$, so that $A_1 \not\mathbin{\overset{|}{\circ}} T|N$. Thus there is a suboperator C of $T|N$ and a suboperator D of A_1 such that $C \cong D$. Let C act on the nonzero space $M (\leqq N)$. Thus by lemma 1,

$$M \subset Eig(D;T) \subset Eig(A_1;T).$$

This is a contradiction to $M \subset N$. Q. E. D.

3. Algebraic implications

For $T \in \mathcal{B}(H)$, let $\Sigma(T)$, $\Sigma_0(T)$ and $\Sigma_{ess}(T)$, respectively, be the reducing operator spectrum, reducing operator eigen spectrum, and the essential reducing operator spectrum of T. Let $C^*(T)$ ($W^*(T)$) be the C^*-algebra (von Neumann algebra, resp.) generated by T and I in $\mathcal{B}(H)$.

PROPOSITION 4. *If $T \in \mathcal{B}(H)$ then $C^*(T) \cap \mathcal{K}(H) = \{0\}$ if and only if $\Sigma_0(T) \subset \Sigma_{ess}(T)$.*

Proof. ($\Leftarrow$) Let $\Sigma_0(T) \subset \Sigma_{ess}(T)$. Assume contrary that $C^*(T) \cap \mathcal{K}(H) \neq \{0\}$. By Corollary 2.9 [4] p. 334, there is an operator $A \in \Sigma_0(T) \sim \Sigma_{ess}(T)$, a contradiction.

($\Rightarrow$) Assume that $C^*(T) \cap \mathcal{K}(H) = \{0\}$. Let $A \in \Sigma_0(T)$. We find an irreducible represenion Π of $C^*(T)$ such that $\Pi(T) = A$. By (ii) of Corllary 1.4 [4] p. 331, we see that $A = \Pi(T) \in \Sigma_{ess}(T)$, as desired. *Q. E. D.*

By using corollary 2.7[4] p. 334, we can prove the following proposition easily.

PROPOSITION 5. *If* $T\in\mathcal{B}(H)$, *then* $W^*(T)\cap\mathcal{K}(H)=\{0\}$ *if and only if* $\Sigma_0(T)=\{A\in\Sigma_0(T): mult(A:T)=\infty\}$.

COROLLARY. *If* $\Sigma_0(T)=\phi$, *then* $W^*(T)\cap\mathcal{K}(H)=\{0\}$.

PROPOSITION 6. (*Spectral Inclusion*). *Let* $S\in\mathcal{B}(K)$ *and* $T\in\mathcal{B}(H)$. *Then the following are equivalent.*

(i) $S\in\Sigma(T)$.

(ii) $\Sigma(S)\subset\Sigma(T)$.

(iii) *There is a represention* Π *of* $C^*(T)$ *onto* $C^*(S)$ *such that* $\Pi(T)=S$, $\Pi(I)=I$ *and* $\mathrm{rank}\Pi(B)\leqq\mathrm{rank}(B)$, *for all* $B\in C^*(T)$.

Proof. We apply Theorem 3.3[4] p. 336, in the obvious way.

LEMMA 2 (*Prof. Hadwin informed to the author.*)

$$\Sigma_{ess}(T)=\{A: \text{There is a unital represention } \Pi: C^*(T)\to C^*(A) \text{ such that } C^*(T)\cap\mathcal{K}(H)\subset ker(\Pi)\}.$$

PROPOSITION 7. *Let* $S\in\mathcal{B}(K)$ *and* $T\in\mathcal{B}(H)$. *Then* (i) $\Sigma_{ess}(S)\subset\Sigma_{ess}(T)$ *if and only if there exists a *-homomorphism* ρ: $C^*(T)/C^*(T)\cap\mathcal{K}(H)\to C^*(S)/C^*(S)\cap\mathcal{K}(K)$, *sending the coset* $[T]=T+C^*(T)\cap\mathcal{K}(H)$ *to* $[S]$ *and* $[I]$ *to* $[I]$.

(ii) $\Sigma_{ess}(S)=\Sigma_{ess}(T)$ *if and only if there exists a *-isomorphism* Π *of* $C^*(T)/C^*(T)\cap\mathcal{K}(H)$ *with* $C^*(S)/C^*(S)\cap\mathcal{K}(K)$, *sending* $[T]$ *to* $[S]$ *and* $[I]$ *to* $[I]$.

Proof (i) We apply lemma 2 in the evident way.

(ii) comes from (i). *Q. E. D.*

Addendum. As an analogue to Theorem 2.8. (p. 334 [4]), we have the following proposition.

PROPOSITION 8. *Let* $A\in\mathcal{B}(K)$ *and* $T\in\mathcal{B}(H)$, *where* H *is infinite dimensional. Then the following conditions are equivalent.*

(i) $A\in\Sigma_0(T)$ *and* $mult(A;T)<\infty$.

(ii) *There is an irreducible *-representation* Π *of* $W^*(T)$ *into* $\mathcal{B}(K)$ *such that* $\Pi(T)=A$ *and* $W^*(T)\cap\mathcal{K}(H)\not\subset ker(\Pi)$.

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