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SCHOTTKY-LANDAU PROPERTY AND HYPERBOLICITY OF COMPLEX MANIFOLDS

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This paper is based on an idea of P. A. Griffiths [3] and M. H. Kwack [4]. The primary aim is to give a characterization of hyperbolic manifolds in terms of Schottky-Landau property. In addition, the author gives an elementary proof of R. Brody's result (cf. [2]) as an application of this characterization.

1. Preliminaries

Let M be a complex manifold of dimension n and $T(M)$ its tangent bundle. A *differential pseudometric* is an upper semicontinuous function F_M: $T(M)\to \boldsymbol{R}$ satisfying

(1) $F_M(z,v)\geq 0$, for any $(z,v)\in T(M)$,

and

(2) $F_M(z,rv)=|r|F_M(z,v)$ for any $r\in \boldsymbol{C}$,

where $\boldsymbol{R}$ and $\boldsymbol{C}$ are the fields of real numbers and of complex numbers, respectively. The *integrated form* of F_M is given by, for all $x,y\in M$,

$$d_F(x,y)=\inf\int_\gamma F_M(z,dz)=\inf\int_0^1 F_M(\gamma(t),\gamma'(t))dt,$$

where the infimum is taken over all piecewise C^1 curves joining x and y in M. A well-known example of a differential pseudometric is a *Kobayashi pseudometric* which is defined by

$$K_M(z,v)=\{|t|: f\in H(D,M),\ f(0)=z,\ f'(0)t=v\}$$

where $H(D,M)$ is the set of holomorphic mappings of the unit disc D in the complex plane $\boldsymbol{C}$ into M. The upper semicontinuity of K_M is proved by H. L. Royden (cf. [5]). A complex manifold is *F_M-hyperbolic* if each point in M has a neighborhood U and admits a positive number m_U depending only on U satisfying:

$$F_M(z,v)\geq m_U\|v\|,$$

for all $z\in U$ and $v\in T_z(M)$. M is said to be *hyperbolic* if it is K_M-hyper-

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bolic.

Given a pair of points (z, w) in M, we choose points $z=z_0, z_1, \cdots, z_k=w$ of M, points $a_1, a_2, \cdots, a_k; b_1, \cdots, b_k$ of D and holomorphic mappings $f_1, \cdots, f_k$ of D into M such that $f_i(a_i)=z_{i-1}$ and $f_i(b_i)=z_i$. The *Kobayashi pseudometric* is defined by

$$d_M(z, w)=\inf \sum_{i=1}^{k} \rho_D(a_i, b_i),$$

where the infimum is taken over all possible choices of points and functions as above and where ρ_D is the Poincaré metric on the unit disc. The following theorems show the relationships between the above terminologies.

THEOREM 1.1. (*H. L. Royden*). *d_M is the integrated form of K_M.*

THEOREM 1.2. (*H. L. Royden, T. J. Barth*). *For a complex manifold M, the following are equivalent:*

(1) *M is hyperbolic.*

(2) *d_M is a proper distance.*

(3) *d_M induces the standard topology on M.*

2. Lemmas

We shall denote by $D(0, r)$ the open disc about origin with radius r in the complex plane. The following two lemmas generalize the lemmas given by M. H. Kwack [4].

LEMMA 2.1. *Let $\{f_k\}$ be a sequence of holomorphic mappings from $D=D(0,1)$ into a hyperbolic manifold M which converges uniformly to an f on each compact subset of D. Further assume that $f_k(0) \in W$ for all indices k and for a fixed open and relatively compact subset W contained in a coordinate neighborhood U in M. Then $df_k(0)$ converges to $df(0)$.*

Proof. By Theorem 1.2., there exist positive numbers r_0, r_1 and r_2 such that

(1) $0<r_0<r_0+r_1<r_2$, and

(2) $\{p \in M : d_M(W, p)<r_2\} \subset U$.

Let t_0 with $0<t_0<1$ satisfy that

$$f(D(0, t_0)) \subset \{p \in M;\ d_M(W, p)<r_0\}.$$

Choose a positve integer K such that for any $k>K$ and for all $x \in D(0, t_0)$,

$$d_M(W, f_k(x)) \leq d_M(W, f(x))+d_M(f_k(x), f(x)) \leq r_0+r_1<r_2,$$

and hence $f_k(\bar{D}(0, t_0)) \subset U$. So the result follows from the Weierstrass' theorem.

LEMMA 2.2. *Let $\{f_k\}$ be a sequence of holomorphic mappings of D into a hyperbolic manifold M such that $f_k(0)\in W$ for all $k=1, 2, \cdots$ and for a fixed relatively compact open subset W of a relatively compact coordinate neighborhood U in M.*

Then, there is a positive real number $t<1$ such that on $D(0, t)$ a subsequence of $\{f_k\}$ converges uniformly to a holomorphic mapping $f: D(0, t)\to M$.

Proof. Note that M is locally compact Hausdorff space. Let $\varphi=(w_1, \cdots, w_n)$ be a coordinate function on U. Since $\bar{W}\cap U^c=\phi$ (empty set), where $U^c=M-U$, choose a positive number t' such that $Q=\{p\in M : d_M(W, P) <t'\}$ becomes a relatively compact subset of U by virtue of Theorem 1.2. Since all the f_k are distance decreasing with respect to d_M, we have

$$f_k(\{p\in D : \rho_D(0, p)<t'\})\subset Q.$$

Note that $\varphi(Q)$ is a bounded subset of $\boldsymbol{C}^n$. Thus by the Montel's theorem, we can choose $t\leq t'$ such that a subsequence of $\{f_k\}$ converging uniformly on $D(0, t)$ to a holomorphic mapping $f: D(0, t)\to M$.

3. Main theorem

THEOREM 3.1. *A complex manifold M is hyperbolic if and only if it has the Schottky-Landau propety, i.e., given a point $z_0\in M$ and a constant $a>0$, there is a neighborhood W of z_0 in M satisfying the following property: for any holomorphic mapping $f: D(0, r) \to M$ with $f(0)\in W$ and $\|df(0)\|\geq a$, there is a constant $R=R(a, W)>0$ depending only on a and W such that $r\leq R$.*

Proof. Assume that M is hyperbolic. Let $z_0\in M$, $a>0$ be given. Choose a relatively compact coordinate neighborhood U about z_0 and a relatively compact open subset W of U containing z_0. Then we claim that this W does the job. Suppose that M doesnot have a Schottky-Landau property, then there is a sequence $\{r_k\}$ of positive real numbers tending to infinity and a sequence of holomorphic mappings $f_k: D(0, r_k)\to M$ with $f_k(0)\in W$ and $\|df_k(0)\|\geq a$. Let $h_k(z)=f_k(r_k z)$, then $h_k: D\to M$ satisfies the properties: $h_k(0)\in W$, $h_k\in H(D, M)$ and $\|dh_k(0)\|\geq ar_k$. Therefore, by Lemma 2.1., $\{h_k\}$ cannot have any subsequence converging uniformly on a neighborhood of 0. But this contradicts Lemma 2.2.

Conversely, assume that M has the Schottky-Landau property. Let $z_0\in M$ be given and let $a=1$. Then a neighborhood W of z_0 in M which forms the Schottky-Landau property. We claim first that, given $(z, v)\in T(M)$, for every $f\in H(D, M)$ with $f(0)=z\in W$ and $df(0)\eta=v$, for some $\eta\in\boldsymbol{C}$,

$$(*) \qquad \|df(0)\|\leq\delta_W,$$

for some δ_W depending only on M and W. Assume the contrary, then for an arbitrarily given $B>0$, there is a holomorphic mapping $f:D\to M$ such that $f(0)=z\in W$, $df(0)\eta=v$ and $\|df(0)\|>B$. Define $h:D(0,B)\to M$ by $h(z)=f(z/B)$. Then h is holomorphic and $\|dh(0)\|=(1/B)\|df(0)\|>1$. Since B can be arbitrarily large, this contradicts the Schottky-Landau property. Therefore there is a constant δ_W satisfying (*).

Hence, given $(z,v)\in T(M)$, $z\in W$, for any $\eta\in C$ such that there is an $f\in H(D,M)$ satisfying that $f(0)=z$ and $df(0)\eta=v$, we have

$$\delta_W|\eta|\geq\|df(0)\|\,|\eta|\geqslant\|v\|$$

or

$$|\eta|\geq m_W\|v\|,$$

where $m_W=1/\delta_W$. This completes the proof.

4. Application to compact cases

Using the device of Theorem 3.1., we will give an elementary proof of the following

THEOREM 4.1. (*R. Brody*). *Let M be a compact complex manifold. Then the following are equivalent:*

(1) *M is hyperbolic.*

(2) $\sup\{\|df(0)\|:f\in H(D,M)\}<\infty$.

(3) *M admits no complex line*, i.e., *there is no nonconstant holomorphic mapping from the complex plane into M.*

Proof. (1)$\Rightarrow$(2). By Theorem 3.1., for any $z_0\in M$, there is a neighborhood W of z_0 in M such that, for any $f\in H(D(0,r),M)$ with $f(0)\in W$ and $\|df(0)\|\geq 1$, there is a constant $R=R(W,1)$, depending only on W, such that $r\leq R$. It follows that every $f\in H(D,M)$ with $f(0)\in W$ satisfies $\|df(0)\|\leq R$. Since M is compact, (2) holds.

(2)$\Rightarrow$(1). Suppose that M is not hyperbolic. Then by Theorem 3.1., there is a point $z_0\in M$ such that there is a sequence of holomorphic mappings $f_k:D(0,r_k)\to M$ satisfying $f_k(0)\in W$ for any coordinate nighborhood W of z_0 and $\|df_k(0)\|\geq 1$, where $\{r_k\}$ is a sequence of positive real numbers tending to infinity. Let $g_k(z)=f_k(r_kz)$, then $g_k\in H(D,M)$ and $\|dg_k(0)\|\geq r_k$ so that $\sup\{\|df(0)\|:f\in H(D,M)\}=\infty$.

(1)$\Rightarrow$(3). Obvious.

(3)$\Rightarrow$(1). Suppose that M is not hyperbolic. Then as above, there exists a sequence of holomorphic mappings $f_k:D(0,r_k)\to M$ such that $f_k(0)\in W$ and $\|df_k(0)\|\geq 1$ for a fixed relatively compact open subset W of a relatively compact coordinate neighborhood in M, and such that r_k tends to infinity as k becomes larger. Note that $H(D(0,r_k),M)$ forms a normal family, since M is compact. Assume $\{r_k\}$ is an increasing sequence with $r_1 \geq 1$.

Let $\{f_{1,i}\}$ be a subsequence of $\{f_k\}$ converging uniformly on all compact subsets of $D(0,r_1)$. For each $n=2,3,4,\cdots$, we choose recursively a subsequence $\{f_{n,k}\}$ of $\{f_{n-1,k}\}$ converging uniformly on all compact subsets of $D(0,r_n)$.

Now choose the diagonal sequence $\{f_{k,k}\}$ and let f be the limit function of the sequence of holomorphic mappings $f_{k,k}$ restricted on the unit open disc. We claim that this f can be extended to a holomorphic mapping $F: C \to M$, which will complete the proof.

Let β be an arbitrary point in $\boldsymbol{C}$. Then there is a natural number p such that $|\beta| < r_p$. Let F_p be the limit function of the sequcence

$$f_{p,p}|_{D(0,r_p)},\ f_{p+1,p+1}|_{D(0,r_p)},\ \cdots.$$

Of course, F_p is holomorphic on $D(0,r_p)$, especially at β, and $F_p|_D = f$.

Thus by the Monodromy theorem we get the result.

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