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INFINITESIMAL VARIATIONS OF SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD

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0. Introduction

Infinitesimal variations of submanifolds of Riemannian and general metric manifolds have been studied by Davies [1], Dienes [2], Hayden [3], Schouten and van Kampen [4] and one of the present authors [5], [6].

Recently, Ki, Okumura and one of the present authors [7] studied infinitesimal variations of invariant submanifolds of a Kaehlerian manifold, and the present authors [8] studied those of anti-invariant submanifolds.

The main purpose of the present paper is to study infinitesimal variations of generic and CR submanifolds of a Kaehlerian manifold.

In §1, we quote some formulas in the theory of submanifolds of a Kaehlerian manifold and in §2 we define and study invariant, anti-invariant, generic and CR submanifolds.

In §3, we obtain rather general formulas for infinitesimal variations of submanifolds of a Kaehlerian manifold and in the last §4, we study invariant, anti-invariant, generic and CR variations.

1. Submanifolds of a Kaehlerian manifold

Let M^{2m} be a real $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$, and $F_i{}^h$ and g_{ji} the almost complex structure tensor and the almost Hermitian tensor of M^{2m} respectively, where, here and in the sequel, the indices $h, i, j, k\ldots$ run over the range $\{1', 2', \ldots, (2m)'\}$. Then we have

(1.1) $$F_j{}^t F_t{}^h = -\delta_j^h,$$

(1.2) $$F_j{}^t F_i{}^s g_{ts} = g_{ji}$$

and

(1.3) $$\nabla_j F_i{}^h = 0,$$

where ∇_j denotes the operator of covariant differentiation with respect to g_{ji}.

Let M^n be an n-dimensional Riemannian manifold covered by a system of

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coordinate neighboroods $\{V; y^a\}$ and g_{cb} the metric tensor of M^n, where, here and in the sequel, the indices $a, b, c, \dots$ run over the range $\{1, 2, \dots, n\}$. We assume that M^n is isometrically immersed in M^{2m} by the immersion $i: M^n \longrightarrow M^{2m}$ and identify $i(M^n)$ with M^n. We represent the immersion $i: M^n \longrightarrow M^{2m}$ locally by

(1.4) $$x^h = x^h(y^a)$$

and put

(1.5) $$B_b{}^h = \partial_b x^h \quad (\partial_b = \partial/\partial y^b).$$

These $B_b{}^h$ are n linearly independant vectors tangent to the submanifold M^n. Since the immersion is isometric, we have

(1.6) $$g_{cb} = g_{ji} B_c{}^j B_b{}^i.$$

We denote by $C_y{}^h$ $2m-n$ mutually orthogonal unit normals to M^n, where, here and in the sequel, the indices x, y, z run over the range $\{n+1, \dots, 2m\}$. Then the equations of Gauss are given by

(1.7) $$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h$$

and those of Weingarten by

(1.8) $$\nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h,$$

where ∇_c denotes the operator of van der Waerden–Bortolotti covariant differentiation along M^n and the second fundamental tensors $h_{cb}{}^x$ and $h_c{}^b{}_y$ are related by

(1.9) $$h_c{}^a{}_y = h_{cby} g^{ba} = h_{cb}{}^z g^{ba} g_{zy},$$

g^{ba} being contravariant components of g_{ba} and g_{zy} the covariant components of the metric tensor of the normal bundle.

Now decomposing $F_i{}^h B_b{}^i$ and $F_i{}^h C_y{}^i$ into tangential and normal parts respectively, we have equations of the form

(1.10) $$F_i{}^h B_b{}^i = f_b{}^a B_a{}^h - f_b{}^x C_x{}^h$$

and

(1.11) $$F_i{}^h C_y{}^i = f_y{}^a B_a{}^h + f_y{}^x C_x{}^h.$$

Since $F_{ji} = -F_{ij}$ where $F_{ji} = F_j{}^t g_{ti}$, we have

(1.12) $$f_{by} = f_{yb},$$

where $f_{by} = f_b{}^z g_{zy}$ and $f_{yb} = f_y{}^c g_{cb}$.

Applying F to the both sides of (1.10) and (1.11), using (1.10) and (1.11) and comparing the tangential and normal parts, we find

(1.13) $$f_b{}^e f_e{}^a - f_b{}^z f_z{}^a = -\delta_b^a,$$

(1.14) $$f_b{}^e f_e{}^x + f_b{}^z f_z{}^x = 0,$$

(1.15) $$f_y{}^e f_e{}^a + f_y{}^z f_z{}^a = 0,$$

(1.16) $$-f_y{}^e f_e{}^x + f_y{}^z f_z{}^x = -\delta_y{}^x.$$

Differentiating (1.10) and (1.11) covariantly along M^n, using (1.10) and (1.11), and comparing tangential and normal parts, we find

(1.17) $$\nabla_c f_b{}^a = h_{cb}{}^x f_x{}^a = h_c{}^a{}_z f_b{}^z,$$

(1.18) $$\nabla_c f_b{}^x = h_{ce}{}^x f_b{}^e - h_{cb}{}^z f_z{}^x,$$

(1.19) $$\nabla_c f_y{}^a = -h_c{}^e{}_y f_e{}^a + h_c{}^a{}_z f_y{}^z,$$

(1.20) $$\nabla_c f_y{}^x = h_c{}^e{}_y f_e{}^x - h_{ce}{}^x f_y{}^e.$$

2. Invariant, anti-invariant, generic and *CR* submanifolds

When the tangent space of M^n is invariant under the action of F, the submanifold M^n is said to be *invariant* or *complex* in M^{2m}. A necessary and sufficient condition for M^n to be invariant is that

(2.1) $$f_b{}^x = 0$$

in (1.10).

When the transform of the tangent space of M^n by F is always normal to M^n, the submanifold M^n is said to be *anti-invariant* or *totally real* in M^{2m} [9]. A necessary and sufficient condition for M^n to be anti-invariant is that

(2.2) $$f_b{}^a = 0$$

in (1.10).

When the transform of the norml space of M^n by F is always tangent to M^n, the submanifold M^n is said to be *generic* in M^{2m} [10]. A necessary and sufficient condition for M^n to be generic is that

(2.3) $$f_y{}^x = 0$$

in (1.11).

When there exist complementary distributions L and M in the tangent sepace of the submanifold M^n and L is invariant under the action of F and M is transformed into a space normal to M^n, the submanifold M^n is called a *CR submanifold* [11].

We denote by $l_b{}^a$ and $m_b{}^a$ the projection operators on L and M respectively. Then we have

(2.4) $$l^2 = l, \quad m^2 = m, \quad lm = ml = 0, \quad l + m = 1.$$

First of all, we have from (1.10)

$$F_i{}^h (B_b{}^i l_c{}^b) = (f_b{}^a l_c{}^b) B_a{}^h - (f_b{}^x l_c{}^b) C_x{}^h,$$

from which, the distribution L being invariant under the action of F, we have

(2.5) $$m_e{}^a f_b{}^e l_c{}^b = 0$$
and

(2.6) $$f_b{}^x l_c{}^b = 0.$$

We also have from (1.10)

$$F_i{}^h (B_b{}^i m_c{}^b) = (f_b{}^a m_c{}^b) B_a{}^h - (f_b{}^x m_c{}^b) C_x{}^h,$$

from which, the distribution M being anti-invariant under the action of F, we have $f_b{}^a m_c{}^b = 0$ and consequently

(2.7) $$f_b{}^a l_c{}^b = f_c{}^a.$$

Thus transvecting (1.14) with $l_c{}^b$ and using (2.6) and (2.7), we find

(2.8) $$f_b{}^e f_e{}^x = 0$$

and consequently

(2.9) $$f_b{}^z f_z{}^x = 0.$$

Conversely, suppose that (2.8) is satisfied. Then we have, from (1.13)

$$f_c{}^b f_b{}^e f_e{}^a + f_c{}^a = 0,$$

which shows that $f_b{}^a$ defines an f-structure. Thus, if we put

$$l_b{}^a = -f_b{}^e f_e{}^a, \quad m_b{}^a = f_b{}^e f_e{}^a + \delta_b{}^a$$

we can easily see that l and m are complementary projection operators defining distributions L and M respectively.

We can verify also that l and m thus defined satisfy

$$m_e{}^a f_b{}^e = 0, \quad f_b{}^x l_c{}^b = 0$$

because of (2.8). Thus we have from (1.10)

$$F_i{}^h (B_b{}^i l_c{}^b) = (f_b{}^a l_c{}^b) B_a{}^h,$$

which shows that L is invariant under the action of F because of $m_a{}^e f_b{}^a l_c{}^b = 0$. We also have from (1.10)

$$F_i{}^h (B_b{}^i m_c{}^b) = -(f_b{}^x m_c{}^b) C_x{}^h,$$

because of $f_b{}^a m_c{}^b = 0$, which shows that M is anti-invariant under the action of F. Thus we have

PROPOSITION 2.1. *A necessary and sufficient condition for a submanifold M^n in M^{2m} to be a CR submanifold is that $f_b{}^e f_e{}^x = 0$.*

3. Infinitesimal variations of submanifolds

We now consider an infinitesimal variation

(3.1) $$\bar{x}^h = x^h + \xi^h(y)\varepsilon$$

of a submanifod M^{2n} of a Kaehlerian manifold M^{2m}, where $\xi^h(y)$ is a vector field of M^{2m} defined along M^n and ε is an infinitesimal. We then have

(3.2) $$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon,$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are n linearly independent vectors tangent to the varied submanifold. We displace $\bar{B}_b{}^h$ parallelly from the varied point $(\bar{x}^h)$ to the original point (x^h). We then obtain

(3.3) $$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon,$$

neglecting the terms of order higher than one with respect to ε. In the sequel, we always neglect terms of order higher than one with respect to ε. Thus if we put

(3.4) $$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h,$$

we have

(3.5) $$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon.$$

If we put

(3.6) $$\xi^h = \xi^a B_a{}^h + \xi^x C_x{}^h,$$

we find

(3.7) $$\nabla_b \xi^h = (\nabla_b \xi^a - h_b{}^a{}_x \xi^x) B_a{}^h + (\nabla_b \xi^x + h_{ba}{}^x \xi^a) C_x{}^h.$$

When the tangent space of the varied submanifold at the varied point $(\bar{x}^h)$ is parallel to the tangent space of the original submanifold at the original point (x^h), the infinitesimal variation (3.1) is said to be *parallel*.

From (3.5) and (3.7) we have

PROPOSITION 3.1. *A necessary and sufficient condition for the infinitesimal variation* (3.1) *to be parallel is*

(3.8) $$\nabla_b \xi^x + h_{ba}{}^x \xi^a = 0.$$

We next consider infinitesimal variations of the unit normals $C_y{}^h$. We denote by $\bar{C}_y{}^h$ $2m-n$ mutually orthogonal unit normals to the varied submanifold and by $\tilde{C}_y{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by the parallel displacement of $\bar{C}_y{}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

(3.9) $$\tilde{C}_y{}^h = \bar{C}_y{}^h + \Gamma_{ji}{}^h(x + \xi\varepsilon)\xi^j \bar{C}_y{}^i \varepsilon,$$

where $\Gamma_{ji}{}^h$ are Christoffel symbols fromed with g_{ji}.

We put

(3.10) $$\delta C_y{}^h = \tilde{C}_y{}^h - C_y{}^h$$

and assume that $\delta C_y{}^h$ is of the form

(3.11) $$\delta C_y{}^h = (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon.$$

Then we have from (3.9), (3.10) and (3.11),

(3.12) $$\bar{C}_y{}^h = C_y{}^h - \Gamma_{ji}{}^h \xi^j C_y{}^i \varepsilon + (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h).$$

Now, applying the operator δ to $g_{ji}B_b{}^jC_y{}^i=0$ and using $\delta g_{ji}=0$, (3.5) and (3.11), we find $(\nabla_b\xi_y+h_{bay}\xi^a)+\eta_{yb}=0$, where $\xi_y=g_{yx}\xi^x$ and $\eta_{yb}=\eta_y{}^c g_{cb}$, from which

$$\eta_y{}^a=-(\nabla^a\xi_y+h_e{}^a{}_y\xi^e)\varepsilon, \tag{3.13}$$

where $\nabla^a=g^{ae}\nabla_e$ Also, applying δ to $g_{ji}C_x{}^jC_y{}^i=g_{xy}=\delta_{xy}$, we find

$$\eta_{yx}+\eta_{xy}=0, \tag{3.14}$$

where $\eta_{yx}=\eta_y{}^z g_{zx}$.

We now compute the variations of $f_b{}^a, f_b{}^x, f_y{}^a$ and $f_y{}^x$ appearing in (1.10) and (1.11). First of all, we put

$$F_i{}^h(x+\xi\varepsilon)\bar{B}_b{}^i=(f_b{}^a+\delta f_b{}^a)\bar{B}_a{}^h-(f_b{}^x+\delta f_b{}^x)\bar{C}_x{}^h.$$

Substituting (3.2) and (3.12) in this equation, using $\nabla_jF_i{}^h=0$ and comparing the tangential and normal parts, we obtain

$$\delta f_b{}^a=[(\nabla_b\xi^e-h_b{}^e{}_x\xi^x)f_e{}^a-f_b{}^e(\nabla_e\xi^a-h_e{}^a{}_x\xi^x)+(\nabla_b\xi^x+h_{be}{}^x\xi^e)f_x{}^a-f_b{}^y(\nabla^a\xi_y+h_e{}^a{}_y\xi^e)]\varepsilon \tag{3.15}$$

and

$$\delta f_b{}^x=[f_b{}^e(\nabla_e\xi^x+h_{ed}{}^x\xi^d)+(\nabla_b\xi^e-h_b{}^e{}_y\xi^y)f_e{}^x-(\nabla_b\xi^y+h_{be}{}^y\xi^e)f_y{}^x-f_b{}^y\eta_y{}^x]\varepsilon. \tag{3.16}$$

We next put

$$F_i{}^h(x+\xi\varepsilon)\bar{C}_y{}^i=(f_y{}^a+\delta f_y{}^a)\bar{B}_a{}^h+(f_y{}^z+\delta f_y{}^z)\bar{C}_z{}^h.$$

Then by a similar computation as above, we find

$$\delta f_y{}^a=[-f_y{}^e(\nabla_e\xi^a-h_e{}^a{}_x\xi^x)-(\nabla^e\xi_y+h_d{}^e{}_y\xi^d)f_e{}^a+\eta_y{}^zf_z{}^a+f_y{}^z(\nabla^a\xi_z+h_d{}^a{}_z\xi^d)]\varepsilon \tag{3.17}$$

and

$$\delta f_y{}^x=[-f_y{}^e(\nabla_e\xi^x+h_{ed}{}^x\xi^d)+(\nabla^e\xi_y+h_d{}^e{}_y\xi^d)f_e{}^x+\eta_y{}^zf_z{}^x-f_y{}^z\eta_z{}^x]\varepsilon. \tag{3.18}$$

4. Infinitesimal variations of invariant, anti-invariant, generic and *CR* submanifolds

Suppose that the submanifold M^n is invariant. Then we have $f_b{}^x=0$ and (3.16) becomes

$$\delta f_b{}^x=[f_b{}^e(\nabla_e\xi^x+h_{ed}{}^x\xi^d)-(\nabla_b\xi^y+h_{be}{}^y\xi^e)f_y{}^x]\varepsilon. \tag{4.1}$$

An infinitesimal variation which carries an invariant submanifold into an invariant one is said to be ***invariant***. From (4.1), we have

PROPOSITION 4.1. *A necessary and sufficient condition for an infinitesimal variation* (3.1) *to be invariant is that*

$$f_b{}^e(\nabla_e\xi^x+h_{ed}{}^x\xi^d)=(\nabla_b\xi^y+h_{be}{}^y\xi^e)f_y{}^x. \tag{4.2}$$

From Propositions 3.1 and 4.1 we have

PROPOSITION 4.2. *A parallel variation is an invariant variation.*

Suppose that the submanifold M^n is anti-invariant. Then we have $f_b{}^a=0$ and (3.15) becomes

(4.3) $$\delta f_b{}^a=[(\nabla_b\xi^x+h_{be}{}^x\xi^e)f_x{}^a-f_b{}^y(\nabla^a\xi_y+h_e{}^a{}_y\xi^e)]\varepsilon.$$

An infinitesimal variation which carries an anti-invariant submanifold into an anti-invariant one is said to be *anti-invariant*. From (4.3), we have

PROPOSITION 4.3. *A necessary and sufficient condition for an infinitesimal variation* (3.1) *to be anti-invariant is that*

(4.4) $$(\nabla_b\xi^x+h_{be}{}^x\xi^e)f_x{}^a=f_b{}^y(\nabla^a\xi_y+h_e{}^a{}_y\xi^e).$$

From Propositions 3.1 and 4.3, we have

PROPOSITION 4.4. *A parallel variation is an anti-invariant variation.*

Suppose now that the submanifold M^n is generic. Then we have $f_y{}^x=0$ and (3.18) becomes

(4.5) $$\delta f_y{}^x=[-f_y{}^e(\nabla_e\xi^x+h_{ed}{}^x\xi^d)+(\nabla^e\xi_y+h_d{}^e{}_y\xi^d)f_e{}^x]\varepsilon.$$

An infiinitesimal vavriation which carries a generic submanifold into a generic one is said to be *generic*. From (4.5), we have

PROPOSITION 4.5. *A necessary and sufficient condition for an infinitesimal variation* (3.1) *to be generic is that*

(4.6) $$f_y{}^e(\nabla_e\xi^x+h_{ed}{}^x\xi^d)=(\nabla^e\xi_y+h_d{}^e{}_y\xi^d)f_e{}^x.$$

From Propositions 3.1 and 4.5, we have

PROPOSITION 4.6. *A parallel variation is generic.*

Finally suppose that the submanifold M^n is a CR submanifold. Then we have $f_b{}^e f_e{}^x=0$. Substituting (3.15) and (3.16) into

$$\delta(f_b{}^e f_e{}^x)=(\delta f_b{}^e)f_e{}^x+f_b{}^e(\delta f_e{}^x),$$

we find

$$\delta(f_b{}^e f_e{}^x)=[(\nabla_b\xi^y+h_{bd}{}^y\xi^d)f_y{}^e f_e{}^x-f_b{}^y(\nabla^e\xi_y+h_d{}^e{}_y\xi^d)f_e{}^x$$
$$+f_b{}^e f_e{}^d(\nabla_d\xi^x+h_{dc}{}^x\xi^c)-f_b{}^e(\nabla_e\xi^y+h_{ed}{}^y\xi^d)f_y{}^x]\varepsilon,$$

from which, using (1.13) and (1.16),

(4.7) $$\delta(f_b{}^e f_e{}^x)=f_b{}^y[f_y{}^e(\nabla_e\xi^x+h_{ed}{}^x\xi^d)-(\nabla^e\xi_y+h_d{}^e{}_y\xi^d)f_e{}^x]\varepsilon$$
$$+[(\nabla_b\xi^z+h_{bd}{}^z\xi^d)f_z{}^y-f_b{}^e(\nabla_e\xi^y+h_{ed}{}^y\xi^d)]f_y{}^x\varepsilon.$$

An infinitesimal variation which carries a CR submanifold into a CR submanifold is called a *CR variation*. From (4.7), we have

PROPOSITION 4.7. *A necessary and sufficient condition for an infinitesimal variation* (3.1) *to be CR variation is that*

$$
(4.8) \qquad f_b{}^y[f_y{}^e(\nabla_e\xi^x + h_{ed}{}^x\xi^d) - (\nabla^e\xi_y + h_d{}^e{}_y\xi^d)f_e{}^x] + [(\nabla_b\xi^z + h_{bd}{}^z\xi^d)f_z{}^y - f_b{}^e(\nabla_e\xi^y + h_{ed}{}^y\xi^d)]f_y{}^x = 0.
$$

From Propositions 3.1 and 4.7, we have

PROPOSITION 4.8. *A parallel variation is a CR variation.*

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