J. Korean Math. Soc.
Vol. 16, No. 2, 1980

PERIODIC MAPS ON PRODUCT 3-MANIFOLDS WHICH ARE ISOTOPIC TO THE IDENTITY

KYUNG WHAN KWUN[1] AND FRANK A. RAYMOND[1]

1. Introduction

Let F be a closed surface not homeomorphic to the 2-sphere or the projective plane. Let h be a periodic homeomorphism of $F\times S^1$ onto itself of a prime period p such that h is isotopic to the identity. Two such h_1 and h_2 are called equivalent if there are a homeomorphism $g: F\times S^1 \to F\times S^1$ and an integer i, $0<i<p$, such that $h_2{}^i=g^{-1}h_1g$. Our result is the following.

THEOREM. *With F, h and p as above,* (1) *if F is not a Klein bottle, or F is a Klein bottle and p is odd, then h has no fixed point and* (2) *if h has no fixed point, h is equivalent to $1_F\times\beta$, where β generates the standard free Z_p action of S^1.*

Simple examples show that Part (1) is false for $F=$a Klein bottle and $p=2$.

2. The proof of Part (1)

Suppose h has a fixed point $x_0\in X=F\times S^1$. Let $q: R^3\to X$ be the universal covering map and $y_0\in q^{-1}(x_0)$. Let $\tilde{h}: (R^3, y_0)\to(R^3, y_0)$ be the periodic homeomorphism of period p such that $q\tilde{h}=hq$. First, $h^{\#}: \pi_1(X, x_0)\to\pi_1(X, x_0)$ is an inner automorphism such that $(h^{\#})^p=$identity. Since there is no nontrivial inner automorphism of $\pi_1(X, x_0)$ of a finite order if F is not a Klein bottle and no nontrivial inner automorphism of an odd order if F is a Klein bottle, it follows that $h_{\#}=$identity. Then it further follows that $\tilde{h}$ fixes $q^{-1}(x_0)$ pointwise. Therefore if $C\subset X$ is the component of the fixed point set of h containing x_0, then $q^{-1}(C)=\tilde{C}$ is pointwise fixed under $\tilde{h}$. Since $\tilde{C}$ is both open and closed in the fixed point set $\tilde{U}$ of $\tilde{h}$ and $\tilde{U}$ is acyclic, $\tilde{C}=\tilde{U}$. Then C and X are both $K(\pi_1(X), 1)$ spaces, $C=X$ and $h=$identity. This contradiction proves Part (1).

Received August 14, 1979
[1] Supported in part by National Science Foundation.

3. The proof of Part(2) for $F \neq a$ Klein bottle

Since the conclusion for $F=S^1\times S^1$ follows from a result [1] of Hempel, assume $F\neq S^1\times S^1$.

Now use q for the orbit map $F\times S^1\to Y$ of the free Z_p action generated by h. Consider the exact sequence

$$1 \longrightarrow \pi_1(X, x_0) \xrightarrow{q_\#} \pi_1(Y, y_0) \xrightarrow{\alpha} Z_p \longrightarrow 0,$$

where $x_0\in X$ and $y_0=q(x_0)$. Choose $g\in\alpha^{-1}(h)$, where h is regarded as a generator of Z_p, the group of covering transformations. Identify $\pi_1(X, x_0)$ with its image under $q_\#$. Then the conjugation of $\pi_1(X, x_0)$ by g is an inner automorphism of $\pi_1(X, x_0)$. Hence g can be chosen to commute with every element of $\pi_1(X, x_0)$. In this case g^p lies in the center of $\pi_1(X, x_0)$, a cyclic group $\pi_1(S^1)$ generated by, say t, where $\pi_1(X, x_0)$ is identified as before with $\pi_1(F)\times\pi_1(S^1)$.

Suppose $g^p=t^k$. Since $\pi_1(Y)$ has no nontrivial element of a finite order, $p\nmid k$. There exist integers x and y such that $px+ky=1$. Then

$$t=t^{px}\cdot t^{ky}=t^{px}\cdot g^{py}=(t^x\cdot g^y)^p.$$

By replacing h by h^y, we may assume $(t^x g)^p=t$. Further replacing g by $t^x g$, we have $g^p=t$. Then $\pi_1(Y, y_0)\simeq\pi_1(F)\times Z$ and $q_\#$ can be considered as $1_{\pi_1(F)}\times r$, where $r(t)=g^p$, g generating Z. Since Y is irreducible, it follows from the fibering theorem [2] of Stallings, $Y\approx F\times S^1$ and considering the covering transformations of q, Part (2) follows in this case.

4. The proof of Part(2) for $F=a$ Klein bottle and $p=2$

Let $q : (F\times S^1, x_0)\to(Y, y_0)$ be the orbit map of Z_2 action generated by h. Again we identify $\pi_1(X, x_0)$ with $q_\#(\pi_1(X, x_0))$. Explicitly write $\pi_1(X, x_0)$ $=\pi_1(F)\times\pi_1(S^1), \pi_1(F)=|a, b : bab^{-1}=a^{-1}|$. Consider the exact sequence

$$1 \to \pi_1(X, x_0) \xrightarrow{q_\#} \pi_1(Y, y_0) \xrightarrow{\alpha} Z_2\to 0.$$

Choose $g\in\alpha^{-1}(h)$ so that g commutes with every element of $\pi_1(X, x_0)$. g^2 is in the center of $\pi_1(X, x_0)$ and $g^2=b^{2k}t^r$, where t generates $\pi_1(S^1)$. r is odd as $\pi_1(Y, y_0)$ has no element of a finite order. If $r=2m+1$, replace g by gt^{-m} and assume $g^2=b^{2k}t$. Then $\pi_1(Y, y_0)=\pi_1(F)\times G$, where G is the cyclic group generated by g. Consider $\pi_1(X, x_0)$ as the direct sum of $\pi_1(F)$ with the cyclic group generated by $b^{2k}t$. By Stallings again $Y\approx F\times S^1$ and $q_\#$ represents the product $(1_F\times\beta)_\#$, β the double covering of S^1 over S^1. This time of course the $\pi_1(S^1)$ portion of $\pi_1(X, x_0)$ is generated by $b^{2k}t$. The automorphism of $\pi_1(X, x_0)$ sending a to a, b to b and t to $b^{2k}t$ can be seen to be induced by a homeomorphism of X onto itself.

5. The proof of Part(2) for F=a Klein bottle and p odd

Again consider $q:(X, x_0)\to(Y, y_0)$ and the exact sequence

$$1\to\pi_1(X, x_0)\to\pi_1(Y, y_0)\to Z_p\to 0.$$

Use the same notations as in Section 3. Then $g^p=b^{2k}t^r$.

Case 1. $(p, r)=1$. There exist integers x and y such that $px+ry=1$. Then $t=t^{px}\cdot t^{ry}=t^{px}(g^p b^{-2k})^y$ and $(g^y t^x)^p=b^{2ky}t$. Replace h by h^y and assume $(gt^x)^p=b^{2ky}t=b^{2k'}t$. Finally replace g by gt^x and assume $g^p=b^{2k'}t$. Consider $\pi_1(X, x_0)$ as the direct sum of $\pi_1(F)$ and the cyclic group generated by $b^{2k'}t$ and $\pi_1(Y, y_0)$ as the direct sum of $\pi_1(F)$ and the cyclic group generated by g. Again we see that q is equivalent to $1_F\times\beta$, β the p-fold covering of S^1. Part(2) for the case now follows.

Case 2. $r=mp$ for some integer m. Replace g by gt^{-m} and get $g^p=b^{2k}$. A trick similar to one in Section 3 allows us to assume $k=1$. Consider $\pi_1(Y, y_0)$ as the direct sum of $\pi_1(S^1)$ with

$$G=|a, b, g:\ ag=ga,\ \ bg=gb,\ \ b^{-1}ab=a^{-1},\ \ g^p=b^2|.$$

G is isomorphic to

$$\bar{G}=|x, y: y^{-1}xy=x^{-1}|$$

under the isomorphism η sending a to x, g to y^2 and b to y^p. η^{-1} sends x to a, y to bg^{-m}, $p=2m+1$. Then the subgroup of G generated by a and b corresponds to the subgroup of $\bar{G}$ generated by x and y^p.

Then again $Y\approx F\times S^1$ and q is equivalent to $\gamma\times 1_{S^1}: F\times S^1\to F\times S^1$, where $\gamma: F\to F$ is the p-fold covering whose group of covering transformations is generated by h described below.

Consider the Klein bottle as the quotient space of $S^1\times R$, $S^1=\{z\in C\,|\,|z|=1\}$, under the identification $(z, t)=(\bar{z}, t+1)$. The equivalence class of (z, t) is denoted by $[z, t]$. $h([z, t])=\left[z, t+\frac{2}{p}\right]$.

The conclusion. There remains to show that $1_F\times\beta$ is equivalent to $\gamma\times 1_{S^1}$. Consider $F\times S^1$ as the quotient of $S^1\times R\times R$ under the identification $(z, t, s)=(\bar{z}, t+1, s)$, where $(z, t, s)=(z, t, s+1)$. The class of (z, t, s) is denoted by $[z, t, s]$. Under this notation, we wish to show h_1, h_2 are equivalent, where $h_1[z, t, s]=\left[z, t, s+\frac{1}{p}\right]$ and $h_2[z, t, s]=\left[z, t-\frac{p-1}{p},\ s\right]$.

Consider $g: F\times S^1\to F\times S^1$ given by $g[z, t, s]=[z, pt+(p-1)s, t+s]$. It is easy to check that g is a well-defined homeomorphism and $g^{-1}h_1 g=h_2$. This completes the proof.

References

1. John Hempel, *Free cyclic actions on* $S^1 \times S^1 \times S^1$, Proc. Amer. Math. Soc. **48** (1975), 221–227.
2. John Stallings, *On fibering certain 3-manifolds, Topology of 3-manifolds*, Prentice Hall (1962), 95–100.

Seoul National University and Michigan State University,
Seoul National University and University of Michigan