韓國數學敎育學會誌 <數學敎育>
1980. 4. Vol. XVIII No. 1

Journal of the Korea Sciety Mathematical Education
April. 1980. Vol. XVIII No. 1

A Generalization of S. P. Singh's T-invariant Point Theorem to Approximation Theory

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Abstract

In [3], an extension of B. Brosowski s T-invariant Point Theorem is given where the linearity of the function and the convexity of the set are relaxed. In this paper, our main purpose is to generalize S. P. Singh's T-invariant Point Theorem to Approximation Theory.

1. Introduction and definitions

Let X be a normed linear space and let S be a subset of X. A self-mapping T of S is said to be non-expansive if $\| Tx-Ty \| \leq \| x-y \|$ for each x, y in S. If $\| Tx-Ty \| < \| x-y \|$ when $x \neq y$, Then T is said to be contractive. Clearly, non-expansive mappings contain all contractive mappings as a proper subclass, and they form a proper subclass of the collection of all continuous mappings. A subset S of X is said to be starshaped if there exists a point x_0 in S such that $tx+(1-t)x_0$ in S whenever $t \in [0, 1]$, $x \in S$. Of course, the star-shaped subsets of X include the convex subsets of X as a proper subclass. In [1], B. Brosowski obtains a result concerning the existence of T-invariant points for contractive linear selfmapping of a compact convex subset of X in Approximation Theory and, in [3], S. P. Singh extends B. Brosowski's T-invariant Point Theorem when the linearity of the function and the convexity of the set are relaxed, In this paper, our main theorem contains as a special case a result of S. P. Singh [3].

2. The main theorem

B. Brosowski has proved the following theorem.

Theorem 1 ([1]) Let T be a contractive linear operator on a normed linear space. Let C be a T-invariant subset of X and x a T-invariant point. If the set of best C-approximants to x is nonempty compact convex, then it contains a T-invariant point.

S. P. Singh [3] has proved a theorem similar to B. Brosowski's T-invariant Point Theorem when T is not a linear operator and the set of best C-approximants to x is not necessarily a convex set.

We are now ready to give the main theorem.

Thoerem 2 Let T be a non-expansive self-mapping on a normed linear space. Let C be a T-invariant subset of X, x a T-invariant point and D the set of best C-approximants to x. Suppose there exists a continuous function f; $[0,1]\times D \longrightarrow D$ which satisfies

(1) $f(1,x)=x$ for every x in D,

(2) there exists a self-mapping, ϕ, of $(0,1)$ such that for every x,y in D and for all t in $(0,1)$,

$$\|f(t,x)-f(t,y)\| \leq \phi(t)\|x-y\|.$$

If D is compact, then it contains a T-invariant point.

Proof Let D be the set of best C-approximants to x. Then T maps D into D. In fact, if $y\in D$, then $\|Ty-Tx\| = \|Ty-Tx\| \leq \|y-x\|$, therefore $Ty\in D$. For each $n=1, 2, \cdots$, let $k_n=n/n+1$ and define $T_n; D\longrightarrow D$ by $T_n x=f(k_n,\ Tx)$ for all $x\in D$. Then, since T maps D into D, T_n also maps D into D for each n. Also, we have, for all x,y in D,

$$\begin{aligned}\|T_n x-T_n y\| &= \|f(k_n,\ Tx)-f(k_n,\ T)\| \\ &\leq \phi(k_n)\|Tx-Ty\| \\ &\leq \phi(k_n)\|x-y\|,\end{aligned}$$

so that each T_n is a contractive mapping on D. Then, since D is compact, each T_n has a unique fixed point, say $x_n\in D$(Edelstein's Fixed Point Theorem [2]), i.e., $T_n x_n=x_n$ for each n. Since D is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\longrightarrow \bar{x}\in D$. Since $T_{n_j}x_{n_j}=x_{n_j}$, we have $T_{n_j}x_{n_j}\longrightarrow \bar{x}\in D$. But T is continuous (since T is non-expansive), and so $Tx_{n_j}\longrightarrow T\bar{x}$. By the continuity of f,

$$T_{n_j}x_{n_j}=f(k_{n_j},\ Tx_{n_j})\to f(1, T\bar{x})=T\bar{x}$$

as $j\to\infty$. It follows that $T\bar{x}=\bar{x}$. Therefore D contains a T-invariant point.
Consequently, Theore 2 yields as a corollary a T-invariant point theorem for non-expansive self-mappings of a normed linear space X which was proved by S.P. Singh for compact star-shaped subsets of X.

Corollary 3 ([3]) Let T be a contractive operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If the set D of best C-approximants to x is nonempty compact star-shaped, then it contains T-invariant point.

Proof Let T be a non-expansive self-mapping of D (as in the proof of Theorem 2). Since D is star-shaped, take a star-center x_0 in D such that $tx+(1-t)x_0\in D$ and define f; $[0,1]\times D\to D$ by $f(t,x)=tx+(1-t)x_0$ for all x in D and for all t in $[0,1]$. Then $f(t,x)$ is continuous in t and x and for all x in D, $f(1,x)=x$ holds. Also, we easily checks that for all x,y in D and for all t in $[0,1]$,

$$\|f(t,x)-f(t,y)\| \leq t\|x-y\|$$

so that we can take $\phi(t)=t$ for all t in $(0,1)$, hence the all conditions are satisfied. Therefore D contains a T-invariant point.

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