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Regular Ideals in an algebra $C_c^\infty(\Omega)$

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1. Introduction.

Let Ω be non-empty open subset of R^n.
Let C_c^∞ (Ω) be the set of all infinitely smooth complex-valued centiouous functions on Ω with compact support. Then $C_c^\infty(\Omega)$ forms a ring under the addition and multiplication defined by the formulas

$$(f+g)(x)=f(x)+g(x), \text{ and } (fg)(x)=f(x)g(x).$$

Furthermore, if the scalar multiplication is defined by the relation $(\alpha f)(x)=\alpha f(x)$ for any scalar α, then it is easy to see that $C_c^\infty(\Omega)$ becomes a commutative algebra. The present note is a study of the regular ideals in the algebra $C_c^\infty(\Omega)$.

2. Regular ideals in an algebra $C_c^\infty(\Omega)$.

Definition. Let A be an algebra. An element $x\in A$ is said to have a left (right) quasi-inverse if there exists some $y\in A$ such that $y\circ x=y+x-yx=0$ $(x\circ y=x+y-xy=0)$, and x is side to have a quasi-inverse if there exists some $y\in A$ such that $y\circ x=x\circ y=0$. If $x\in A$ has a quasi-inverse, then x is said to be quasi-reqular, or quasi-invertible, and x is said to be quasi-singular if it is not quasi-regular.

Theorem 1. If f is a $C_c^\infty(\Omega)$-function such that $\inf_{\omega\in\Omega}|1-f(\omega)|>0$, then

$$\{-h+hf\,|\,h\in C_c^\infty(\Omega)\}=C_c^\infty(\Omega).$$

Proof. The function g defined by $g(\omega)=\dfrac{f(\omega)}{f(\omega)-1}$ is a $C_c^\infty(\Omega)$-function and hence belongs to $C_c^\infty(\Omega)$. Obviously $f\circ g=0$, so that f is quasi-regular in $C_c^\infty(\Omega)$.

And $f=-g+gf\in\{-h+hf\,|\,h\in C_c^\infty(\Omega)\}$. Clearly $\{-h+hf\,|\,h\in C_c^\infty(\Omega)\}$ is an ideal in $C_c^\infty(\Omega)$, and so for any $u\in C_c^\infty(\Omega)$ we have

$$u=(u-uf)+nf\in\{-h+hf\,|\,h\in C_c^\infty(\Omega)\},$$

that is, $\{-h+hf\,|\,h\in C_c^\infty(\Omega)\}=C_c^\infty(\Omega)$.

Lemma. Let p be a point of Ω, and let $Mp=\{f\in C_c^\infty(\Omega)\,|\,f(p)=0\}$.

Then Mp is a maximal ideal of $C_c^\infty(\Omega)$.

Proof. It is obvious that Mp is an ideal. In fact. if $f\in Mp$ and $g\in Mp$, i.e. $f(p)=0$,

$g(p)=0$, then $(\alpha f+\beta g)(p)=\alpha f(p)+\beta g(p)=0$ where α and β are scalars, and $(hf)(p)=h(p)f(p)=0$ for any function $h\in C_c^\infty(\Omega)$. To show that Mp is maximal, let $g\notin Mp$. For any function $f\in C_c^\infty(\Omega)$, we shall prove that $f=gf_1+f_2u$ for some $u\in Mp$ and $f_i\in C_c^\infty(\Omega)$ $(i=1,2)$. Let the support of g be K. then $\mathring{K}$, the interior of K, is not empty since $p\in\mathring{K}$. Let us choose two compact sets K_1 and K_2 such that $g\neq 0$ on $\mathring{K}_2$, $p\in K_1\subset K_2$ and $K_2\subset\mathring{K}$. We can find $u_1\in C_c^\infty(\Omega)$ such that $u_1\equiv 1$ on K_1 and supp $u_1\subset K_2$. Let g^* be defined by $g^*=g^{-1}$ on K_2 and $g^*=0$ on $\Omega-K$ Then $g^*u_1\in C_c^\infty(K_2)$ and $gg^*u_1\equiv u_1$. Let $u_2\in C_c^\infty(\Omega)$ such that $u_2\equiv 1$ on $CK_2\cap$ supp f, supp $u_2\subset CK_1$ and $u_1+u_2=1$ on supp f. Then $f=fgg^*u_1+fu_2$, which completes our proof.

Definition. Let A be an algebra. A left (right, two-sided) ideal I in A is said to be regular if there exists some $u\in A$ such that $xu-x\in I(ux-x\in I,\ xu-x\in I$ and $ux-x\in I)$, x in A.

Theorem 2. Let p be a point of Ω. Then the maximal ideal Mp in $C_c^\infty(\Omega)$ is regular.

Proof. Since there exists $g\in C_c^\infty(\Omega)$ such that $g(p)=1$,

$(fg-f)(p)=f(p)g(p)-f(p)=f(p)-f(p)=0$ for $f\in C_c^\infty(\Omega)$.

That is, $fg-f\in Mp$.

Theorem 3. Let Mp be a maximal regular ideal in the algebra $C_c^\infty(\Omega)$. If $f\in C_c^\infty(\Omega)$ and $\inf_{\omega\in\Omega}|1-f(\omega)|>0$, then there exists some $g\in C_c^\infty(\Omega)$ such that $f\circ g\in Mp$.

Proof. Let $g=\frac{f}{f-1}$. Then $g\in C_c^\infty(\Omega)$ and f has a quasi-inverse g. That is, $f\circ g=f+g-fg=0$. This gives that $f\circ g\in Mp$.

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