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AN APPROXIMATE EVALUATION OF SIN X

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Introduction

Recently O.R. Ainsworth and C.K. Liu developed a formula concerning to a binormal expansion (cf. [1]); namely,

(1) $\left(1+\sum_{m=1}^{\infty}(-1)^m a_m x^m\right)^{-p}=1+\sum_{q=1}^{\infty}\frac{b_q x^q}{q!}$

Here p stands for an arbitrary real number and b_q is the constant which equals to the p-th power of the following determinants;

(2) $b_q=p$

a_1	1	0	0	0	0	0	0	········	O
$2a_2$	$(p+1)a_1$	2	0	0	0	0	0	········	O
$3a_3$	$(2p+1)a_2$	$(p+2)a_1$	3	0	0	0	0	········	O
$4a_4$	$(3p+1)a_3$	$(2p+2)a_2$	$(p+3)a_1$	4	0	0	0	········	O
$5a_5$	$(4p+1)a_4$	$(3p+2)a_3$	$(2p+3)a_2$	$(p+4)a_1$	5	0	0	········	O
$6a_6$	$(5p+1)a_5$	$(4p+2)a_4$	$(3p+3)a_3$	$(2p+4)a_2$	$(p+5)a_1$	6	0	········	O
$7a_7$	$(6p+1)a_6$	$(5p+2)a_5$	$(4p+3)a_4$	$(3p+4)a_3$	$(2p+5)a_2$	$(p+6)a_1$	7	········	O
$8a_8$	$(7p+1)a_7$	$(6p+2)a_6$	$(5p+3)a_5$	$(4p+4)a_4$	$(3p+5)a_3$	$(2p+6)a_2$	$(p+7)a_1$	········	O
$9a_9$	$(8p+1)a_8$	$(7p+2)a_7$	$(6p+3)a_6$	$(5p+4)a_5$	$(4p+5)a_4$	$(3p+6)a_3$	$(2p+7)a_2$	········	O
$10a_{10}$	$(9p+1)a_9$	$(8p+2)a_8$	$(7p+3)a_7$	$(6p+4)a_6$	$(5p+5)a_5$	$(4p+6)a_4$	$(3p+7)a_3$	········	O
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮		
qa_q	$[(q-1)p+1]a_{q-1}$	$[(q-2)p+2]a_{q-2}$	$[(q-3)p+3)a_{q-3}$	$[(q-4)p+4]a_{q-4}$	$[(q-5)p+5]a_{q-5}$	$[(q-6)p+6]a_{q-6}$	$[(q-7)p+7]a_{q-7}$		$(p+q-1)a_1$

It turns out that the formula (1) can be usefully applied to the numerical evaluations of the real powers of many important functions. In this paper, we shall evluate an approximate value of $\sin^{\frac{1}{3}}x$ and $\int_0^{x}\sin^{\frac{1}{3}}x\,dx$ using the formula (1). It should be emphasized that similar method can be applied for the evaluation of other functions which can be expanded into a Taylor series.

Main results

By the Taylor series expansion formula, we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots\cdots$$

$$= x\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m \cos\frac{m\pi}{2}}{(m+1)!} x^m\right].$$

Therefore

$$(3)\quad \sin^{\frac{1}{3}} x = x^{\frac{1}{3}}\left[1 + \sum_{m=1}^{\infty} (-1)^m a_m x^m\right]^{\frac{1}{3}}$$

$$= \left(1 + \sum_{q=1}^{\infty} \frac{b_q x^q}{q!}\right) x^{\frac{1}{3}}$$

where $a_m = \frac{\cos\frac{m\pi}{2}}{(m+1)!}$, $\qquad p = -\frac{1}{3}$.

Since $a=0$, $a=-\frac{1}{3!}$, $a=0$, $a=\frac{1}{5!}$, $a=0$, $a=-\frac{1}{7!}$, $a=0$,

$a=\frac{1}{9!}$, $a=0$, $a=-\frac{1}{11!}$, $a=0$, $a=\frac{1}{13!}$, $a=0$, $a=-\frac{1}{15!}$, ……

taking $p=-\frac{1}{3}$, we get by (2)

$$b_q = -\frac{1}{3}\left|\begin{array}{ccccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots \\
-\frac{2}{3!} & 0 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots\cdots \\
0 & \frac{1}{3}\left(-\frac{1}{3!}\right) & 0 & 3 & 0 & 0 & 0 & 0 & \cdots\cdots \\
\frac{4}{5!} & 0 & \frac{4}{4}\left(-\frac{1}{3!}\right) & 0 & 4 & 0 & 0 & 0 & \cdots\cdots \\
0 & -\frac{1}{3}\left(\frac{1}{5!}\right) & 0 & \frac{7}{3}\left(-\frac{1}{3!}\right) & 0 & 5 & 0 & 0 & \cdots\cdots \\
-\frac{6}{7!} & 0 & \frac{2}{3}\left(\frac{1}{5!}\right) & 0 & \frac{10}{3}\left(-\frac{1}{3!}\right) & 0 & 6 & 0 & \cdots\cdots \\
0 & -\left(-\frac{1}{7!}\right) & 0 & \frac{5}{3}\left(\frac{1}{5!}\right) & 0 & \frac{13}{3}\left(-\frac{1}{3!}\right) & 0 & 7 & \cdots\cdots \\
\frac{8}{9!} & 0 & 0 & 0 & \frac{8}{3}\left(\frac{1}{5!}\right) & 0 & \frac{16}{3}\left(-\frac{1}{3!}\right) & 0 & \cdots\cdots \\
0 & -\frac{5}{3}\left(\frac{1}{9!}\right) & 0 & -\frac{1}{7!} & 0 & \frac{11}{3}\left(\frac{1}{5!}\right) & 0 & \frac{19}{3}-\left(\frac{1}{3!}\right) & \cdots\cdots \\
-\frac{10}{11!} & 0 & -\frac{2}{3}\left(\frac{1}{9!}\right) & 0 & 2\left(-\frac{1}{7!}\right) & 0 & \frac{14}{3}\left(\frac{1}{5!}\right) & 0 & \cdots\cdots \\
0 & -\frac{8}{3}\left(-\frac{1}{11!}\right) & 0 & \frac{1}{3}\left(\frac{1}{9!}\right) & 0 & 3\left(-\frac{1}{7!}\right) & 0 & \frac{17}{3}\left(\frac{1}{5!}\right) & \cdots\cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}\right.$$

We evaluate b_q's upto b_q as follows;

$$b_1=-\frac{1}{3}\,|\,0\,|=0$$

$$b_2=-\frac{1}{3}\begin{vmatrix} 0 & 1 \\ -\frac{1}{3} & 0 \end{vmatrix}=-\frac{1}{9} \qquad b_3=-\frac{1}{3}\begin{vmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 2 \\ 0 & -\frac{1}{18} & 0 \end{vmatrix}=-\frac{1}{3}\times 0=0$$

$$b_4=-\frac{1}{3}\begin{vmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 2 & 0 \\ 0 & -\frac{1}{18} & 0 & 3 \\ \frac{1}{30} & 0 & -\frac{2}{9} & 0 \end{vmatrix}=-\frac{1}{3}(-3)\begin{vmatrix} 0 & 1 & 0 \\ -\frac{1}{3} & 0 & 2 \\ \frac{1}{30} & 0 & -\frac{2}{9} \end{vmatrix}=\frac{1}{15}-\frac{2}{27}=-\frac{1}{135}$$

$$b_5=-\frac{1}{3}\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 2 & 0 & 0 \\ 0 & -\frac{1}{18} & 0 & 3 & 0 \\ \frac{1}{30} & 0 & -\frac{2}{9} & 0 & 4 \\ 0 & -\frac{1}{360} & 0 & -\frac{7}{18} & 0 \end{vmatrix}=\frac{4}{3}\begin{vmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 2 & 0 \\ 0 & -\frac{1}{18} & 0 & 3 \\ 0 & -\frac{1}{360} & 0 & -\frac{7}{18} \end{vmatrix}=-\frac{8}{3}\begin{vmatrix} 0 & 1 & 0 \\ 0 & -\frac{1}{18} & 3 \\ 0 & -\frac{1}{360} & -\frac{7}{18} \end{vmatrix}=0$$

$$b_6=-\frac{1}{3}\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 2 & 0 & 0 & 0 \\ 0 & -\frac{1}{18} & 0 & 3 & 0 & 0 \\ \frac{1}{30} & 0 & -\frac{2}{9} & 0 & 4 & 0 \\ 0 & -\frac{1}{360} & 0 & -\frac{7}{18} & 0 & 5 \\ -\frac{1}{840} & 0 & \frac{1}{180} & 0 & -\frac{5}{9} & 0 \end{vmatrix}=\frac{1}{3}(-5)\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 2 & 0 & 0 \\ 0 & -\frac{1}{18} & 0 & 3 & 0 \\ \frac{1}{30} & 0 & -\frac{2}{9} & 0 & 4 \\ -\frac{1}{840} & 0 & \frac{1}{180} & 0 & -\frac{5}{9} \end{vmatrix}$$

$$=-5\begin{vmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 2 & 0 \\ \frac{1}{30} & 0 & -\frac{2}{9} & 4 \\ -\frac{1}{840} & 0 & \frac{1}{180} & -\frac{5}{9} \end{vmatrix}=5\begin{vmatrix} -\frac{1}{3} & 2 & 0 \\ \frac{1}{30} & -\frac{2}{9} & 4 \\ -\frac{1}{840} & \frac{1}{180} & -\frac{5}{9} \end{vmatrix}$$

$$=5\left(-\frac{10}{243}-\frac{8}{840}+\frac{4}{540}+\frac{10}{270}\right)=-\frac{53}{1701}$$

Substituting the above values in (3), we have

(4) $\sin^{\frac{1}{3}}x = x^{\frac{1}{3}}(1+0\cdot x-\frac{1}{9}\cdot\frac{1}{2!}x^2+0\cdot x^3-\frac{1}{135}\cdot\frac{1}{4!}x^4+0\cdot x^5-\frac{53}{1701}\cdot\frac{1}{6!}x^6+\cdots\cdots)$

$$=x^{\frac{1}{3}}-\frac{1}{18}x^{\frac{7}{3}}-\frac{1}{9720}x^{\frac{13}{3}}-\frac{53}{1224720}x^{\frac{19}{3}}-\cdots\cdots$$

Convergence of the above infinite series is guarantees by the formula (1) (cf. [1]). To evaluate $\int_0^\pi \sin^{\frac{1}{3}}x dx$, we integrate right hand side of (4) term by term and get

(5) $\int_0^\pi \sin^{\frac{1}{3}}x dx = \int_0^\pi \left(x^{\frac{1}{3}}-\frac{1}{18}x^{\frac{7}{3}}-\frac{1}{9720}x^{\frac{13}{3}}-\frac{53}{1224720}x^{\frac{19}{3}}-\cdots\cdots\right)dx$

$$=\left.\frac{4}{3}x^{\frac{4}{3}}-\frac{1}{18}\cdot\frac{18}{3}x^{\frac{10}{3}}-\frac{1}{9720}\cdot\frac{16}{3}x^{\frac{16}{3}}-\frac{53}{1224720}\cdot\frac{22}{3}x^{\frac{22}{3}}-\cdots\cdots\right]_0^\pi$$

$$=x^{\frac{4}{3}}\left(0.75-\frac{1}{60}x^2-\frac{1}{16\times3240}x^4-\frac{53}{22\times408720}x^6-\cdots\cdots\right)\Big]_0^\pi$$

$$=\pi^{\frac{4}{3}}\left(0.75-\frac{1}{60}\pi^2-\frac{1}{16\times3240}\pi^4-\frac{53}{22\times408720}\pi^6-\cdots\cdots\right)$$

$$=4.6011493(0.75-0.1644934-0.00563709-0.005666652-\cdots\cdots)$$

$$\fallingdotseq 4.6011493\times0.574203$$

$$=2.6419937$$

as an approximate evaluation.

Remark; The author was informed by C. K. Liu that a student at Pusan National University evaluated the same definite integral via another method, called graphical method, and got

(6) $\int_0^\pi \sin^{\frac{1}{3}}x dx = 2.577$

We have 0.064 as a difference between (5) & (6). This difference may be reduced by taking more terms in Taylor series expansion of $\sin^{\frac{1}{3}}x$.

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Reference

1. Ainsworth, O. R.; Liu, C. K., An application of Legendre functions in the inversion of a Hilbert matrix, J. Franklin Inst. 299(1975), 297-299.