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A note on the Riemann Integrability

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1. Introduction

In the elementary calculus we usualy define the Riemann definite integral of a function $f(x)$ as follows:

Let f be a bounded function defined on $[a,b]$ and let $\Delta_n=\{a=x_0, x_1, \ldots x_n=b\}$ be any subdivision of $[a,b]$ and ξ_i be any point in the i-th subinterval $[x_{i-1}, x_i]$ where $i=1,2,\cdots,$ n. Let us consider the Riemann sum $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$, and we allow n, the number of subintervals, to become infinite in such a way that all the lengths Δx_i approach zero. If such a sum has a limit, i.e., $\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$ exists, then the function f is said to be Riemann integrable on $[a,b]$, and this limit is called the Riemann integral of the function f over the interval $[a,b]$ and is written by $\int f(x)dx$.

But this definition for the Riemann integral is nonsence because the Riemann sum $\sum_{i=1}^{n} f(\xi_i)\Delta x_i$ could not be treated as a sequence with respect to n nor a function of n. It can be easily seen that the cardinality of the set $\{\Delta_n | \Delta_n$ is a subdivision of $[a,b]$ for a fixed $n\}$ is Ψ_1 and that the cardinality of the set $\{(\xi_1, \cdots, \xi_n)\} | \xi_i \in [x_{i-1}, x_i]\}$ for the fixed n and the fixed subdivision Δ_n is also Ψ_1. Hence for each n there correspond $\Psi_1 \times \Psi_1 = \Psi_1$ values of Riemann sums. Therefore the definition of Riemann integral in terms of the concept of limit should be based on the concept of filter or net.

2. Filters

Definition 1. A *filter* on a set X is a family $\mathscr{F}$ of subsets of X which has the following properties:

i) Every subset of X which contains a member of $\mathscr{F}$ belongs to $\mathscr{F}$.

ii) Every finite intersection of members of $\mathscr{F}$ belongs to $\mathscr{F}$.

iii) $\phi \notin \mathscr{F}$

From the definition of the filter it can be easily seen that the family $N(x)$ of all neighborhoods of a point X in a topologycal space $(X, \mathscr{T})$ is a filter.

Definition 2. Given two filters $\mathscr{F}, \mathscr{F}'$ on the same set X, $\mathscr{F}'$ is said to be *finer* than $\mathscr{F}$, or $\mathscr{F}$ is *coarser* than $\mathscr{F}'$, if $\mathscr{F} \subset \mathscr{F}'$.

Given a family $\mathfrak{A}$ of subsets of a set X, let us consider whether there are any filters on X which contains $\mathfrak{A}$. On the existence of such a filter we have following proposition:

Proposition 1. *A necessary and sufficient condition that there should exist a filter on X containing a family $\mathfrak{A}$ of subsets of X is that no finite subfamily of $\mathfrak{A}$ has an empty intersection.*

Let $\mathfrak{A}$ be the family of subsets of X such that no finite subfamily of $\mathfrak{A}$ has an empty intersection. The coarsest filter $\mathfrak{F}$ which contains $\mathfrak{A}$ is said to be generated by $\mathfrak{A}$, and $\mathfrak{A}$ is said to be a subbase of $\mathfrak{F}$.

Let us consider a family $\mathfrak{B}$ of subsets of X which satisfies the following two conditions:

B_1) The intersection of two members of $\mathfrak{B}$ cotains a member of $\mathfrak{B}$.

B_2) $\mathfrak{B}$ is not empty, and the empty subset of X is not in $\mathfrak{B}$.

Then it can easily seen that the family of all subsets of X which contains a member of $\mathfrak{B}$ is a filter.

Definition 3. A family $\mathfrak{B}$ of subsets of a set X which satisfies the conditions B_1) and B_2) is said to be a *base* of the filter $\mathfrak{F}$ consisting of all subsets of X containing a member of $\mathfrak{B}$. Two filter bases are said to be *equivalent* if they generate the same filter.

Now let us consider the limit of filter in a topological space.

Definition 4. Let $(X, \mathfrak{T})$ be a topological space and $\mathfrak{F}$ a filter on X. A point $x \in X$ is said to be a *limit* of $\mathfrak{F}$, if $\mathfrak{F}$ is finer than the neighborhood filter $N(x)$ *of* x; $\mathfrak{F}$ is also said to converge to x. The point x is also said to be a *limit* of a filter base on X, and $\mathfrak{B}$ is said to *converge* to x, if the filter whose base is $\mathfrak{B}$ converges to x.

3. Definition of Riemann integrals.

Let a bounded function $f: [a, b] \longrightarrow \mathbf{R}$ be given, where $\mathbf{R}$ is a set of all reals with the usual topology, and let $\mathfrak{D}$ be the family of all subdivisions of $[a, b]$. We introduce an order $\prec$ in $\mathfrak{D}$ as follows:

For each pair $(\Delta, \Delta') \in \mathfrak{D} \times \mathfrak{D}$ we define

$$\Delta \prec \Delta' \text{ if and only if } \Delta \subset \Delta'$$

where $\Delta = \{a = x_0, x_1, \cdots, x_n = b\}$ and $\Delta' = \{a = x'_1, x'_2, \cdots, x'_m = b\}$. Then $(\mathfrak{D}, \prec)$ becomes a directed set. For each subdivision $\Delta = \{a = x_0, x_1, \cdots, x_n = b\}$ of $[a, b]$, let R_Δ be the set of all reals,

$$\sum_{i=1}^{n} f(\xi_i) \Delta x_i$$

for all choises $\xi_i \in [x_{i-1}, x_i]$, where $\Delta x_i = x_i - x_{i-1}$. Define $A_\Delta = \cup \{R_{\Delta'} | \Delta \prec \Delta', \ \Delta' \in \mathfrak{D}\}$ for each $\Delta \in \mathfrak{D}$, then we have the following theorem.

Theorem 1. *$\mathfrak{B} = \{A_\Delta | \Delta \in \mathfrak{D}\}$ is a filter base on R.*

Proof. Let $A_\Delta, A_{\Delta'}$ be two members of $\mathfrak{B}$, and let $\Delta'' = \Delta \cup \Delta'$, then $\Delta'' \in \mathfrak{D}$ and $\Delta \prec \Delta''$, $\Delta' \prec \Delta''$. Hence $A_{\Delta''} \subset A_\Delta$ and $A_{\Delta''} \subset A_{\Delta'}$. Therefore we have

$$A_{\Delta''} \subset A_\Delta \cap A_{\Delta'}.$$

That is, $\mathfrak{B}$ satisfies the condition B_1) for the base. It is clear that $\mathfrak{B}$ satisfies the condition

B_2) for the base.

Now we discuss on the Riemann integrability of a bounded function f defined on $[a, b]$ with $P \leqslant f(x) \leqslant Q$ for each $x \in [a, b]$. Let $\Delta = \{a = x_0, x_1, \cdots x_n = b\}$ be any subdivision of $[a, b]$ and let $sup\{f(x) \mid x \in [x_{i-1}, x_i]\} = M_i$, $inf\{f(x) \mid x \in [x_{i-1}, x_i]\} = m_i$ and $\Delta x_i = x_i - x_{i-1}$. Let us consider the following sums

$$U(f, \Delta) = \sum_{i=1}^{n} M_i \Delta x_i, \quad L(f, \Delta) = \sum_{i=1}^{n} m_i \Delta x_i.$$

We then have for each pair (Δ, Δ')

$$P(b-a) \leqslant L(f, \Delta') \leqslant U(f, \Delta) \leqslant Q(b-a).$$

Hence $U(f, \Delta)$ and $L(f, \Delta')$ are bounded and there exsist $inf\{U(f, \Delta) \mid \Delta \in \mathscr{D}\}$ and $sup\{L(f, \Delta') \mid \Delta' \in \mathscr{D}\}$. An usual definition of the Riemann integrability of the function f on $[a, b]$ is as follows:

A bounded function f is Riemann integrable on $[a, b]$ if and only if $\inf_{\Delta}\{U(f, \Delta)\} = \sup_{\Delta}\{L(f, \Delta)\}$.

In the following theorem we see that the Riemann integrability can be characterized by the concept of limit.

Theorem 2. *A bounded function f defined on $[a, b]$ is Riemann integrable on $[a, b]$ if and only if the filter base $\mathscr{B} = \{A_\Delta \mid \Delta \in \mathscr{D}\}$ converges to some point in* **R**.

Proof. ($\Longrightarrow$). Since f is Riemann integrable on $[a, b]$, for each $\varepsilon > 0$, there is $\Delta \in \mathscr{D}$ such that $U(f, \Delta) - L(f, \Delta) < \frac{\varepsilon}{2}$. And for any value $\lambda \in A_\Delta$, $L(f, \Delta) \leqslant \lambda \leqslant U(f, \Delta)$ and $L(f, \Delta) \leqslant I \leqslant U(f, \Delta)$, where $I = \sup_{\Delta}\{L(f, \Delta)\}$. Hence $A_\Delta \subset (I - \varepsilon, I + \varepsilon)$. This means that filter base $\mathscr{B}$ converges to I.

($\Longleftarrow$). Let the filter base converge to a point $I \in$ R. Then for any ε-neighborhood $N_\varepsilon(I)$ of I there exsist a member $A_\Delta \in \mathscr{B}$ such that $A_\Delta \subset N_\varepsilon(I)$. Hence $U(f, \Delta) - L(f, \Delta) < \varepsilon$, that is,

$$\inf_{\Delta}\{U(f, \Delta)\} = \sup_{\Delta}\{L(f, \Delta)\}.$$

Therefore f is Riemann integrable on $[a, b]$.

References.

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