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The Riemann Integral and the Lebesgue Integral

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1. Introduction

From elementary calculus we have a definition of $\int_a^b f(x)dx$ for continous functions f: $[a, b] \longrightarrow \mathbf{R}$, where $[a, b]$ is a closed interval on the set $\mathbf{R}$ of all real numbers. The integration process, as conceived in elementary calculus, is too restrictive in many ways. It would be better to have a definition of an integral applicable to a *larger class of* ***functions***, with a *larger class of sets admissible as the* ***domains*** of definition of the functions.

The Riemann theory of integration was constructed as a generalization of the integration of continuous functions in elementary calculus. But the Riemann theory of integration is inadequate for some of the purposes of higher analysis. It suffer from defects in connection with convergent sequences of functions. It is also too restrictive in that it does not deal adequately with the definition and properties of $\int_E f(x)dx$ if f is an unbounded function or if E is an unbounded set.

Many attempts were made to broaden the class of functions which could be brought under a satisfactory theory of integration. The decisive advance was made by Lebesgue in the first year of 20th century. His study of integration was based on improvement and generalization of the work of Borel on the theory of measure. The Lebesgue theory of integration meets and overcomes many difficulties in the Riemann theory.

The purpose of this paper is to discuss the defects of the Riemann integral and the Lebesgue integral.

In section 2, we will discuss the definition and some properties of the Riemann integral. We also state conditions for a function f on [a, b] to have a primitive function [Theorem 2.4], and a necessary and sufficient condition for the function f to be Riemann-integrable [Theorem 2.3, Theorem 2.2].

In section 3, we will discuss the definition and some important properties of the Lebesgue integral.

Finally, in section 4, we will study the defects of the Lebesgue integral by showing several counterexamples [Theorem 4.1, Theorem 4.2, Theorem 4.3].

2. The Riemann Integral

Let $[a, b]$ be a given closed interval. By a partition of $[a, b]$ we mean a finite set

$$\Delta = \{x_0, x_1, \ldots, x_n\}$$

of numbers such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

Denote the lenth of each subinterval $[x_{i-1}, x_i]$ by Δx_i, that is, let

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, 2, \ldots, n).$$

The maximum of the Δx_i is called the ***norm*** of the partition Δ and is denoted by ρ_Δ. Thus

$$\rho_\Delta = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}.$$

Definition 2.1 Let $f: [a, b] \longrightarrow \mathbf{R}$ be a functon. For each partition $\Delta = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ choose one point ξ_i in each subinterval $[x_{i-1}, x_i]$ such that

$$x_{i-1} \leq \xi_i \leq x_i \quad (i = 1, 2, \ldots, n).$$

and let

$$S_\Delta = \sum_{i=1}^{n} f(\xi_i) \Delta x_i.$$

The sum S_Δ is called a *Riemann sum.*
If there exists a number I such that

$$I = \lim_{\rho_\Delta \to 0} S_\Delta,$$

where varies over the class of all partitions of $[a, b]$, then f is said to be *Riemann-integrable,* and the value I is called the *Riemann integral* of f over $[a, b]$ and is denoted by

$$\int_a^b f(x)dx.$$

Definition 2.2 Let $f: [a, b] \longrightarrow \mathbf{R}$ be a Riemann-integrable function. Then we define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

If a function f is defined at a, then we define

$$\int_a^a f(x)dx = 0.$$

The question of the Riemann-integrability of a function is a delicate one. The following theorem is wellknown (See [4], p. 125).

Theorem 2.1 *If $f: [a, b] \longrightarrow \mathbf{R}$ is a continuous function, then f is Riemann-integrable.*

The next example shows that the converse of Theorem 2.1 does not hold.

Example 1. Let $f: [-1, 1] \longrightarrow R$ be a function defined by

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Then f is not continuous at $x = 0$. But f is Riemann-integrable over $[-1, 1]$. To show this, let

$$\Delta = \{x_0, x_1, \ldots, x_n\}$$

be a partition of $[-1, 1]$ and choose ξ_i such that $x_{i-1} \le \xi_i \le x_i$ for all $i=1, 2, \ldots, u$. Then

$$S_\Delta = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = f(\xi_j)\Delta x_j$$

for some j. Hence

$$\lim_{\rho_\Delta \to 0} S_\Delta = 0.$$

This implies that f is Riemann-integrable and $\int_{-1}^{1} f(x)dx = 0$.

The following theorem gives a necessary condition for a function to be Riemann-integrable.

Theorem 2. *If a function $f: [a, b] \longrightarrow \mathbf{R}$ is Riemann-integrable, then f is bounded.*

Proof Let

$$I = \int_a^b f(x)dx = \lim_{\rho_\Delta \to 0} S_\Delta.$$

Then there exists a partition $\Delta = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$|S_\Delta - I| < 1$$

for all choices of the ξ_i with $x_{i-1} \le \xi_i \le x_i$. For each j let ξ_j be any number in $[x_{j-1}, x_j]$, and let $\xi_i = x_i$ for $i \ne j$. Then we have

$$|f(\xi_j)\Delta x_j + \sum_{i \ne j}^{n} f(x_i)\Delta x_i - I| < 1.$$

If we put

$$M = \sum_{i \ne j}^{n} f(x_i)\Delta x_i,$$

then M is a constant, it follows from the above that

$$\frac{I-1-M}{\Delta x_j} < f(\xi_j) < \frac{I+1+M}{\Delta x_j}.$$

This implies that f is bounded on the subinterval $[x_{j-1}, x_j]$ for each $j=1, 2, \ldots, n$. Hence f is bounded on $[a, b]$.

As we can see in the next example, the converse of Theorem 2.2 is not true.

Example 2. Let $f: [a, b] \longrightarrow \mathbf{R}$ be a function defined by

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational.} \end{cases}$$

Then f is bounded, but it is not Riemann-integrable on $[a, b]$.

In fact, for any partition $\Delta = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ if we choose rational numbers for the ξ_i then

$$S_\Delta = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} \Delta x_i = b-a,$$

and if we choose irrational numbers for the ξ_i then

$$S_\Delta = \sum_{i=1}^{n} f(\xi_i)\Delta x_i = \sum_{i=1}^{n} -\Delta x_i = a-b.$$

Thus the limit of S_Δ as $\rho_\Delta \to 0$ does not exist, and f is not Biemann-integrable.

The following theorem gives a necessary and sufficient condition for a function $f: [a, b]$

$\longrightarrow$**R** to be Riemann-integrable.

Theorem 2.3 *Let $f: [a, b] \longrightarrow \mathbf{R}$ be a bounded function. Then f is Riemann-integrable if and only if the points at which f is discontinuous form a set of measure zero.*

Proof The proof of this theorem can be found in [5].

The contable set of **R** is of measue zero. For example, the set **Q** of all rational numbers is of measure zero. From this fact, we have the following corollary to Theorem 2.3.

Corollary *Let $f: [a, b] \longrightarrow \mathbf{R}$ be a bounded function. If f is discontinous at at most countably many points, then f is Riemann-integrable.*

The following theorem is also wellknown (for the proof, see [4], p. 133).

Theorem 2.4 *Let $f: [a, b] \longrightarrow \mathbf{R}$ be a Riemann-integrable function. Define a function ϕ: $[a, b] \longrightarrow \mathbf{R}$ by*

$$\phi(x) = \int_a^x f(t)dt, \quad a \leq x \leq b.$$

Then the following hold.

(1) *ϕ is a continuous function.*

(2) *If f is continous at x_0, then ϕ is differentiable at x_0 and*

$$\phi'(x_0) = f(x_0).$$

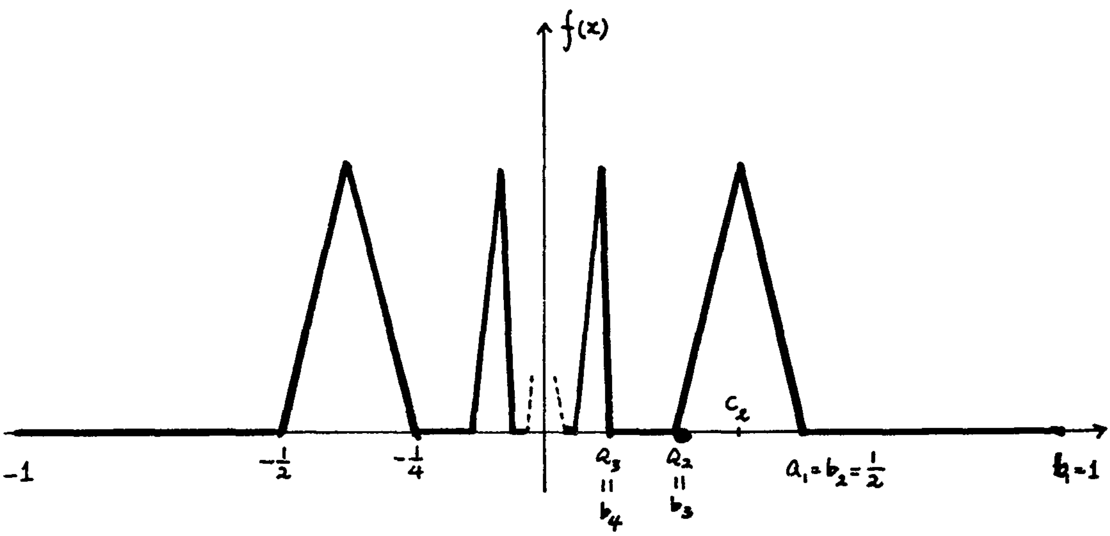
In the above theorem, the continuity of f at x_0 is critical. Consider the following example.

Example 3. For each positive integer n, let

$$a_n = 2^{-n}, \quad b_n = 2^{-(n-1)} \text{ and } c_n = \frac{1}{2}(a_n + b_n).$$

Consider a function $f: [-1, 1] \longrightarrow \mathbf{R}$ defined by

$$f(x) = (x - a_{2n})(c_{2n} - a_{2n})^{-1} \quad \text{for } a_{2n} \leq n \leq c_{2n},$$



$$f(x)=(b_{2n}-x)(b_{2n}-c_{2n})^{-1} \quad \text{for } c_{2n}\leq x\leq b_{2n},$$
$$f(x)=0 \quad \text{for } a_{2n-1}<x<b_{2n-1} \text{ and } x=0,$$
$$f(x)=f(-x) \quad \text{for } -1\leq x<0.$$

Then f is continous everywhere except for $x=0$. Hence f is Riemann-integrable, by Corollary to Theorem. 2.4.

Define a function $\phi:[-1,1]\longrightarrow\mathbf{R}$ by

$$\phi(x)=\int_{-1}^{x}f(t)dt.$$

We will show that ϕ is not differentiable at $x=0$.

If $h=b_{2n}$, then

$$\frac{\phi(h)-\phi(0)}{h}=\frac{1}{h}\int_0^h f(t)dt=\frac{1}{h}\sum_{i=n}^{\infty}2^{-2i-1}$$
$$=\frac{3^{-1}\cdot 2^{-(2n-1)}}{2^{-(2n-1)}}=3^{-1}.$$

If we put $h=b_{2n+1}$, then

$$\frac{\phi(h)-\phi(0)}{h}=\frac{1}{h}\int_0^h f(t)dt=\frac{1}{h}\sum_{i=n}^{\infty}2^{-2i+1}$$
$$\frac{3^{-1}\cdot 2^{-(2n+1)}}{2^{-2n}}=6^{-1}.$$

Thie implies that

$$\lim_{h\to 0}\frac{\phi(h)-\phi(0)}{h}$$

does not exist, and that ϕ is not differentiable at $x=0$.

3. The Lebesgue Integral

In this section we will discuss the definition and some important properties of the Lebesgue integral.

Definition 3.1 Let $f:E\longrightarrow\mathbf{R}$ be a bounded and measurable function where $E=[a,b]$. Let

$$m=\inf f(x) \quad (a\leq x\leq b).$$
$$M=\sup f(x) \quad (a\leq x\leq b).$$

For each partition

$$\Delta=\{y_0,y_1,\dots,y_n\}$$

of the closed interval $[m,M]$, we put

$$E_j=\{x\in E\,|\,y_{j-1}\leq f(x)\leq y_j\}.$$

and

$$s(\Delta)=\sum_{j=1}^{n}y_{j-1}\mu(E_j),\quad S(\Delta)=\sum_{j=1}^{n}y_j\mu(E_j).$$

Let

(1) $\underline{\int_E} fd\mu=\inf s(\Delta)$,

(2) $\overline{\int}_E fd\mu = \sup S(\Delta)$,

where the inf and the sup are taken over all partition Δ of E.

If the left members of (1) and (2) are equal, then we say that f is *Lebesgue-integrable* on E. This common value is called the *Lebesgue integral* of f on E, and is denoted by

$$\int_E fd\mu \text{ or } \int fd\mu.$$

Theorem 3.1 *Let $f:[a,b]\longrightarrow \mathbf{R}$ be a bounded function. If f is measurable, then f is Lebesgue-integrable. Moreover, if f is Riemann-integrable, then f is Lebesgue-integrable and the Riemann integral of f over $[a,b]$ is equal to the Lebesgue integral of f over $[a,b]$.*

Proof The proof can be found in [5].

Any continuous function on $[a,b]$ is Riemann-integrable. Hence any continuous function is also Lebesgue-integrable. The following example shows that the converse of Theorem. 3.1 doesnot hold.

Example 1. Let the function f on $[a,b]$ be defined by

$$f(x)=\begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational.} \end{cases}$$

Then f is not Riemann-integrable (Example 2 in section 2). But $f=g$ a.e., where g is the constant function with value -1 on $[a,b]$, and f is Lebesgue-integrable and $\int fd\mu = a-b$.

The next theorem provides the key to the fundamental convergence theorems of the Lebesgue integral.

Theorem 3.2 (Monotone Convergence Theorem) *Let $\{f_n\}$ be a sequence of bounded and measurable functions on $[a,b]$ such that $0\le f_n \le f_{n+1}$ for all positive integer n.*

If $\{f_n\}$ converges to f, then

$$\int fd\mu = \lim \int f_n d\mu.$$

4. Main Theorems

In this section we will discuss the limitation of the Lebesgue integral by showing counterexamples.

Theorem 4.1 *The product of two Lebesgue-integrable functions on a Borel set may not be Lebesgue-integrable.*

Proof Consider a Borel set $A=(0,1)$, and two functions f and g defined on the set A by

$$f(x)=g(x)=\frac{1}{\sqrt{x}}.$$

Then both f and g are Lebesgue-integrable on A, and

$$\int_A fd\mu = \int_A gd\mu$$

$$=\lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}dx=\lim_{t\to 0^+}\Big[2\sqrt{x}\Big]_t^1=2.$$

The product $f\cdot g : A\longrightarrow \mathbf{R}$ of f and g is defined by

$$(f\cdot g)(x)=f(x)g(x)=\frac{1}{x}.$$

Let $A_n=[(n+1)^{-1}, n^{-1}]$ for each positive integer n. Then the A_n are mutually disjoint, and $A=\bigcup_{n=1}^{\infty}A_n$. Hence

$$\int_A f\cdot g d\mu=\sum_{n=1}^{\infty}\int_{A_n} f\cdot g\ d\mu=\sum_{n=1}^{\infty}\int_{\frac{1}{1+n}}^{\frac{1}{n}}\frac{1}{x}dx$$

$$>\sum_{n=1}^{\infty}n\Big(\frac{1}{n}-\frac{1}{n+1}\Big)=\sum_{n=1}^{\infty}\frac{1}{n+1}.$$

Since the series $\sum_{n=1}^{\infty}\frac{1}{n+1}$ is divergent, it follows that $\int_A f\cdot g d\mu$ does not exist. That is, the product $f\cdot g$ is not Lebesgue-integrable.

Theorem 4.2 *In the Monotone Convergence Theorem (Theorem 3.2), the condition that*

$$0\le f_n\le f_{n+1}$$

can not be removed.

Proof For each positive integer n, define a function $f_n : [0, 1]\longrightarrow \mathbf{R}$ by

$$f_n(x)=\begin{cases} n, & 0<x<\frac{1}{n} \\ 0, & \text{elsewhere.}\end{cases}$$

Then it is easy to see that $\{f_n\}$ converges to $f=0$.

On the other hand,

$$\int f_n\ d\mu=\int_0^1 f(x)dx=1 \text{ and } \int f d\mu=0.$$

Therefore,

$$\int f d\mu=\int \lim f_n\ d\mu=0\ne 1=\lim\int f_n d\mu.$$

The following theorem shows that even if a function on $[a, b]$ is differentiable its derivative f' may not be Lebesgue-integrable.

Theorem 4.3 *There exists a differentiable function* $f: [0, 1]\longrightarrow \mathbf{R}$ *whose derivative* f' *is not Lebesgue-integrable.*

Proof Consider a function $f: [0, 1]\longrightarrow \mathbf{R}$ defined by

$$f(x)=\begin{cases} x^2\cos(\pi x^{-2}), & x\ne 0 \\ 0, & x=0\end{cases}$$

Then f is differentiable and its derivative $f' : [0, 1]\longrightarrow \mathbf{R}$ is given by

$$f'(x)=\begin{cases} 2x\cos(\pi x^{-2})+2\pi x^{-1}\sin(\pi x^{-2}), & x\ne 0 \\ 0, & x=0.\end{cases}$$

Therefore, for $0<a<b\le 1$, the function f' is Lebesgue-integrable over $[a, b]$, and

$$\int_a^b f'(x)dx=b^2\cos(\pi b^{-2})-a^2\cos(\pi a^{-2}).$$

In particular, if we put

$$a_n = \sqrt{2}(4n+1)^{-\frac{1}{2}}, \quad b_n = (2n)^{-\frac{1}{2}},$$

then we have

$$\int_{a_n}^{b_n} f'(x)dx = \frac{1}{2n}.$$

Since the intervals $[a_n, b_n]$ are mutually disjoint and $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$ is contained in the interval $[0, 1]$, we have

$$\int_0^1 |f'(x)|dx \geq \int_A |f'(x)|dx \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \infty.$$

Hence f' is not Lebesgue-integrable, and f' is not Lebesgue-integrable.

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