Nonparametric Estimators of Ratio of Scale Parameters Based on Rank-Like Tests

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ABSTRACT

A class of nonparametric estimators of the ratio of scale parameters is proposed. The estimators are based on the distribution-free rank-like test suggested by Fligner and Killeen (1976). An explicit form of the estimator is the median of the ratios of absolute deviations from the combined sample median. A small-sample Monte Carlo study shows that the proposed estimator is more efficient than the Bhattacharyya (1977) estimator. The proposed estimator is reasonably insensitive to small failures in the assumption of equal medians. A modified estimator is also considered when the medians are unequal.

1. Introduction

Let $X_1,\ldots,X_m$ and $Y_1,\ldots,Y_n$ be two independent random samples from populations with absolutely continuous distribution functions $F((x-\eta_1)/\sigma_1)$ and $F((x-\eta_2)/\sigma_2)$, respectively, where η_1 and η_2 are population medians and σ_1 and σ_2 are scale parameters. We assume that $F'(x)=f(x)$ is symmetric about zero and the two distributions have a common median η, i.e., $\eta_1=\eta_2=\eta$. In this paper we are interested in developing a class of estimators of the ratio of scale parameters $\Delta=\sigma_1/\sigma_2$. The scale estimators

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are based on the test statistics which are distribution-free under an appropriate null hypothesis about Δ.

Assuming that η_1 and η_2 are known, Noether (1967) suggested a procedure to obtain a confidence interval for Δ. Laubscher and Odeh (1976) proposed a technique for obtaining a confidence interval for Δ based on Sukhatme (1957, 1958) test. They also provided the table of critical values of the Sukhatme's statistic.

Bhattacharyya (1977) introduced several Hodges-Lehmann type estimators of Δ, which are scale version of the location estimators proposed by Hodges and Lehmann (1963). The Bhattacharyya estimators are based on Ansari-Bradley (1960), modified Ansari-Bradley, Siegel-Tukey (1960), and modified Sukhatme tests. He used the concept of relevant pairs of observations adjusted for the combined sample median, and the explicit form of the estimators is reduced to the median of the ratios of relevant pairs.

Fligner and Killeen (1976) and Fligner (1979) proposed a class of rank-like test statistics for testing $H_0 : \sigma_1 = \sigma_2$ vs. $H_1 : \sigma_1 > \sigma_2$, which is especially appealing if the populations are symmetric. Fligner and Killeen (1976) investigated some analogs of the Ansari-Bradley, Mood (1954), and Klotz (1962) tests. Each analog has the same Pitman efficiency as its corresponding linear rank statistic, and yet its small-sample power is significantly higher. The estimators considered in this paper are based on the Fligner-Killeen test statistics.

Section 2 deals with the definition and general properties of the proposed estimators. In Sections 3, we consider two estimators which are applicable in practice. When the Wilcoxon scores are used, the explicit form of the corresponding estimator is the median of the ratios of absolute deviations from the combined sample median. If the two medians η_1 and η_2 are significantly different, we may use a modified estimator based on the absolute deviations from each sample median. A small-sample comparison of the proposed nonparametric estimators with the Bhattacharyya and parametric estimators is given in Section 4. The results show that the proposed estimators perform quite well as regards robustness and efficiency when the distributions are heavy-tailed. The proposed estimator which uses the adjusted observations for the combined sample median, is alway superior to the Bhattacharyya estimator in a small-sample Monte Carlo study.

2. Scale Estimators Based on Rank-Like Tests

Let $M(X_1,\ldots, X_m, Y_1,\ldots, Y_n)$ be the combined sample median, which is an estimator of the common population median η. We also let

$$V_i=|X_i-M|, \text{ for } i=1,\ldots, m, \qquad W_j=|Y_j-M|, \text{ for } j=1,\ldots, n, \tag{2.1}$$

and R_i be the rank of V_i in the combined sample of $N=m+n$ $V's$ and $W's$. Fligner and Killeen (1976) proposed the test statistic, for testing

$$H_0 : \sigma_1=\sigma_2,\ \eta_1=\eta_2 \quad \text{vs.} \quad H_1 : \sigma_1>\sigma_2,\ \eta_1=\eta_2,$$

which is defined by

$$T_N=\sum_{i=1}^{m} a_N(R_i) \tag{2.2}$$

where $a_N(\mathrm{i}), i=1,\ldots, N,$ is a vector of scores. Note that T_N is not a rank statistic. It is a rank-like statistic in the sense that the variables that are ranked are not the original observations but functions of them. Under H_0 the joint distribution of N $V's$ and $W's$ are symmetric (exchangeable) in its components, and thus, according to Fligner, Hogg and Killeen (1976) the distribution of T_N in (2.2) is distribution-free under H_0. A good elementary exposition of the distribution-free rank-like tests is contained in Randles and Wolfe (1979, Chapter 11).

We now present a class of two-sample scale estimators based on rank-like tests which is a scale version of the Hodges-Lehmann type location estimators. Following the notations in Hodges and Lehmann (1963), we consider the test statistic of the form

$$h(V,W)=\sum_{i=1}^{m} a_N(R_i) \tag{2.3}$$

where $V=(V_1,\dots, V_m)$ and $W=(W_1,\dots, W_n)$ are vectors of observations defined by (2. 1)

We assume that the test statistic $h(V,W)$ in (2. 3) satisfies the following scale version of the two Hodges-Lehmann conditions:

(A) $h(v,aw)$ is a nonincreasing function of a for all v and w when $a>0$; (2. 4)

(B) when $\Delta=1$, the distribution of $h(V,W)$ is symmetric about a fixed point μ (independent of F).

Assuming the above two conditions, we define a class of estimators $\hat{\Delta}$ of $\Delta=\sigma_1/\sigma_2$ as follows. We let

$$\begin{aligned}\Delta^* &=\text{Sup}\,\{\Delta : h(v, \Delta w)>\mu\},\\ \Delta^{**}&=\text{Inf}\,\{\Delta : h(v, \Delta w)<\mu\},\end{aligned} \tag{2.5}$$

and let

$$\hat{\Delta}=(\Delta^*+\Delta^{**})/2. \tag{2.6}$$

Then for suitable choice of scores we propose $\hat{\Delta}$ as an estimator of Δ. Thus, we actually estimate the ratio of scale parameters of the transformed variables V and W, not the original X and Y.

We here present the regularity properties of $\hat{\Delta}$ without proof in the following theorem, which can be easily shown using the same techniques in Hodges and Lehmann (1963).

Theorem 2. 1. (i) Suppose that (V,W) is a random vector with joint distribution H. Then the distributions of Δ^*, Δ^{**}, and $\hat{\Delta}$ are (absolutely) continuous if H is (absolutely) continuous.

(ii) $\hat{\Delta}$ is scale invariant in the sense that

$$\hat{\Delta}(v, aw)=a^{-1}\,\hat{\Delta}(v, w).$$

(iii) $\hat{\Delta}$ is a median unbiased estimator of the ratio of scale parameters of V and W if

$P_{\Delta=1}\{h(V,W)=\mu\}=0.$

3. Estimator Corresponding to the Wilcoxon Scores Statistic

Using the Wilcoxon scores, the test statistic defined by (2. 2) becomes

$$T_N=\sum_{i=1}^{m} R_i,$$

which has the same distribution as the two-sample Wilcoxon (1945) statistic under H_0. To derive an equivalent test statistic, we define

$$h_1(V,W)=\sum_{i=1}^{m}\sum_{j=1}^{n} I(V_i,W_j) \tag{3.1}$$

where I is a counting function defined as

$$I(v,w)=\begin{cases}1 & \text{if } v>w\\ 0 & \text{otherwise,}\end{cases}$$

and (v,w) is an observed outcome of (V,W). Then $h_1(V,W)$ in (3. 1) is the Mann-Whitney (1947) statistic under H_0. When there are no ties among $V's$ and $W's$, T_N and $h_1(V,W)$ are linearly related by $h_1(V,W)=T_N-m(m+1)/2$. If we set v/w to a sufficiently large number (e.g.,∞) for $w=0$, then $h_1(V,W)$ can be written as

$$h_1(V,W)=\text{number of pairs } (v,w) \text{ with } v/w>1. \tag{3.2}$$

From the definition of $h_1(V,W)$, it can be easily shown that the two conditions given by (2. 4) are satisfied. Moreover, the point of symmetry μ of $h_1(V,W)$ is $mn/2$.

Now, to find an explicit form of the estimator $\hat{\Delta}$, let the ordered values of the mn ratio estimates be denoted by

$$(v/w)_{(1)}<(v/w)_{(2)}<\ldots<(v/w)_{(mn)}.$$

Then it follows that

$$\begin{aligned}\Delta^*&=\text{Sup}\,\{\Delta : h_1(v,\Delta w)>\mu\}\\ &=\text{Sup}\,\{\Delta : \#(v/(\Delta w)>1)>mn/2\}\\ &=\text{Sup}\,\{\Delta : \text{more than } mn/2 \text{ of the } (v/w)\text{'s are greater than } \Delta\}\\ &=\begin{cases}(v/w)_{((mn+1)/2)}, & \text{for } mn \text{ odd}\\ (v/w)_{(mn/2)}\quad , & \text{for } mn \text{ even.}\end{cases}\end{aligned} \tag{3.3}$$

Similarly,

$$\begin{aligned}\Delta^{**} &= \text{Inf}\ \{\Delta : h_1(v, \Delta w) < \mu\} \\ &= \begin{cases} (v/w)_{((mn+1)/2)}, & \text{for } mn \text{ odd} \\ (v/w)_{(mn/2+1)}\ , & \text{for } mn \text{ even.} \end{cases}\end{aligned} \tag{3.4}$$

Combining (3.3), (3.4), and (2.6), we have

$$\hat{\Delta} = \begin{cases} (v/w)_{((mn+1)/2)} & , \text{ for } mn \text{ odd} \\ [(v/w)_{(mn/2)} + (v/w)_{(mn/2+1)}]/2, & \text{for } mn \text{ even.} \end{cases}$$

Thus, $\hat{\Delta}$ is the median of the set of mn ratio estimates v/w and will be denoted by

$$\hat{\Delta} = \text{med}\ (v/w). \tag{3.5}$$

When mn is large, following the techniques in Lehmann (1975) we can obtain $\hat{\Delta}$ in some easier way. In the following example we shall illustrate a shortcut method of obtaining $\hat{\Delta}$.

Example 3.1. Consider the following set of data, taken from Bhattacharyya (1977), which are already adjusted for the combined sample median.

x : −4.7, −2.6, −2.0, −1.3, −0.9, 7.6, 9.7, 9.8

y : −1.9, −1.2, 0.9, 1.9, 4.2, 5.3

Since $mn=48$, the estimate $\hat{\Delta}$ is the average of $(v/w)_{(24)}$ and $(v/w)_{(25)}$, where $v=|x|$ and $w=|y|$. Now, construct a rectangle as shown in Figure 1 and compute some diagonal entries (underlined entries in the figure).

Figure 1. Computation of $\hat{\Delta}$

v \ w	0.9	1.3	2.0	2.6	4.7	7.6	9.7	9.8
0.9	1.00	$\underline{1.44}$	2.22					
1.2		1.08	$\underline{1.67}$	2.16				
1.9			1.05	$\underline{1.37}$	2.47			
1.9				1.37	$\underline{2.47}$			
4.2					1.12	$\underline{1.81}$		
5.3					0.89	1.43	$\underline{1.83}$	

The number of entries≥ 1.37 is equal to $(7+6+5+5+3+3)=29$, where the numbers in the paranthesis correspond to the number of entries ≥ 1.37 in each row. Thus, the smallest rank occupied by 1.37 is $48-29+1=20$. The number of entries ≥ 1.37 and <1.44 is 3 and therefore the rank of 1.44 is 23. Thus, the median of the entries is $(1.67+1.81)/2=1.74$. Note that the Bhattacharyya's estimate, which is the median of ratios of the relevant pairs, is 2.01. (See Section 4 for the definition of relevant pairs.)

We now consider a modified Fligner-Killeen test. When the two population medians η_1 and η_2 are unknown and significently different, the Fligner-Killeen test may be modified by adjusting $X's$ and $Y's$ for each sample median. Let $\hat{\eta}_1$ and $\hat{\eta}_2$ be the sample medians of m $X's$ and n $Y's$, respectively. We also let $V_i'=|X_i-\hat{\eta}_1|$, $i=1,\dots,m$, and $W_j'=|Y_j-\hat{\eta}_2|$, $j=1,\dots,n$. Then a modified Fligner-Killeen test statistic may be defined by

$$h_2(V', W')=\sum_{i=1}^{m} R_i \tag{3.6}$$

where R_i is the rank of V_i' in the combined sample of V' and W'. Notice that $h_2(V', W')$ does not any more have the same distribution as the two-sample Wilcoxon statistic under H_0. But, a natural estimator corresponding to this modified Fligner-Killeen test may be defined by

$$\bar{\Delta}=\text{med }(v'/w') \tag{3.7}$$

which is the median of the set of mn ratios v'/w'. Here again we set v'/w' to a sufficiently large number if $w'=0$.

4. A Small-Sample Monte Carlo Study

In this section we compare the estimators $\hat{\Delta}$ and $\bar{\Delta}$ discussed in Section 3 with the Bhattacharyya and parmetric estimators for small samples. Since the four estimators in Bhattacharyya (1977) are eventually of the same form, we consider only the estimator based on the Ansari-Bradley test.

Let X' and Y' be the X and Y observations, respectively, adjusted by subtracting the combined sample median. Let the scores $a_N(\mathrm{i})$, when $m+n$ is even, be given by

$$\frac{1}{2}(m+n),\ \ldots,\ 2,\ 1,\ 1,\ 2,\ \ldots,\ \frac{1}{2}(m+n) \tag{4.1}$$

If $m+n$ is odd, the scores can be given in a similar way (see Ansari and Bradley, 1960). Then the Ansari-Bradley test statistic can be written by

$$h_3(X',\ Y')=\sum_{i=1}^{m} a_N(R_i) \tag{4.2}$$

where R_i is the rank of X_i' in the adjusted combined sample. Thus $h_3(X', Y')$ in (4.2) is the sum of scores in (4.1) associated with the X-sample in the ordered combined sample of m $X's$ and n $Y's$. Notice that the values of $h_3(X', Y')$ in (4.2) is not altered by subtracting the combined sample median from the original observations.

Let a relevant pair be a pair (x', y') where x' and y' are both positive or both negative. Let the p relevant pairs of adjusted observations be arranged in ascending order of magnitude, and denoted by

$$(x'/y')_{(1)}<(x'/y')_{(2)}<\ldots<(x'/y')_{(p)}.$$

Now, define

$$\Delta^0=\text{Sup}\ \{\Delta : h_3(x',\ \Delta y')>\mu\},$$

$$\Delta_0=\text{Inf}\ \{\Delta : h_3(x',\ \Delta y')<\mu\},$$

where μ is the point of symmetry of $h_3(X', Y')$, i.e., $\mu=mn/4$, and let

$$\tilde{\Delta}=(\Delta^0+\Delta_0)/2.$$

Then $\tilde{\Delta}$ is the Bhattacharyya estimator of Δ based on the Ansari-Bradley test. Applying the same technique in Section 3, the explicit form of the estimator $\tilde{\Delta}$ is given by

$$\tilde{\Delta}=\text{med}(x'/y') \tag{4.3}$$

where (x', y') are all relevant pairs for the adjusted observations.

In the Monte Carlo study, we included as a parametric estimator the square root of the F-ratio, $\sqrt{F}=S_x/S_y$, where S_x^2 and S_y^2 are the usual unbiased estimators for the variances in normal theory. The remainder of this section deals with the results of a comparative study. The nonparametric estimators considered in this study are $\hat{\Delta}$, $\bar{\Delta}$, and $\tilde{\Delta}$, which are defined by (3.5), (3.7), and (4.3), respectively.

In each case 500 sets of two samples with a common sample size $m=n=10$ are generated from the uniform, normal, double exponential, and Cauchy distributions. For each distribution, X and Y populations are considered with different scale and/or location parameters. The uniform and normal random variates were generated by using the subroutines RANDU and GAUSS, respectively, from the IBM Scientific Subroutine Package. The inverse probability integral transformation was used to generate double exponential and Cauchy random samples.

In order to know the insensitivity of the estimators $\tilde{\Delta}$ and $\hat{\Delta}$ to the assumption of equal medians, we investigated the behavior of the estimators when the assmuption of equal medians is violated. Following the definition of Fligner (1979), we define a difference in location between the X and Y distributions in terms of a parameter δ, where

$$\delta=\int_{\eta_1}^{\eta_2} dF((x-\eta_2)/\sigma_2).$$

Thus $|\delta|$ can be interpreted as a fixed amount of probability placed between η_1 and η_2 as measured by the distribution of Y. Through the simuilation study σ_1/σ_2 is set to 2. The values of δ considered in this study are 0 (the case $\eta_1=\eta_2$), 0.2, and 0.3. In the case $\delta=0.1$, the results are very similar to the case $\delta=0$ for each estimator, and thus they are omitted in this report. When $\sigma_2=1$, the values of $\eta_1-\eta_2$ corresponding to $\delta=0.3$ are 0.84, 0.92, and 1.38 for normal, double exponential, and Cauchy distributions, respectively.

Table 1. Empirical Means and Variances* of the Estimators (Sample size $m=n=10$, 500 pairs simulated with $\Delta=2$)

δ	$\check{F}$	$\tilde{\Delta}$	$\hat{\Delta}$	$\bar{\Delta}$
	(a) Uniform Distribution			
0	2.06(0.30)	2.34(1.05)	2.24(0.94)	2.23(1.09)
0.2	2.08(0.28)	2.49(1.74)	2.28(1.08)	2.22(1.02)
0.3	2.09(0.30)	3.53(333.)	2.26(0.96)	2.21(0.97)
	(b) Normal Distribution			
0	2.12(0.55)	2.34(1.32)	2.26(1.15)	2.26(1.29)
0.2	2.08(0.52)	2.30(1.36)	2.19(1.17)	2.19(1.19)
0.3	2.10(0.58)	2.38(2.58)	2.23(1.41)	2.20(1.35)

	(c) Double Exponential Distibution			
0	2.26(1.71)	2.40(2.97)	2.32(1.95)	2.37(2.34)
0.2	2.26(1.29)	2.53(2.85)	2.40(2.43)	2.44(2.40)
0.3	2.37(1.63)	3.06(28.7)	2.48(2.59)	2.41(2.19)
	(d) Cauchy Distribution			
0	8.36(1639.)	2.67(7.82)	2.51(4.09)	2.62(5.20)
0.2	1.05(5029.)	2.86(8.07)	2.68(6.60)	2.76(7.28)
0.3	10.34(3133.)	3.44(27.1)	2.74(5.40)	2.71(5.25)

* Empirical variances appeared in parantheses.

Table 2. Empirical Relative Efficiencies (Inverse Ratio of Empirical Variances)

δ	$(\tilde{\Delta}, \sqrt{F})$	$(\hat{\Delta}, \sqrt{F})$	$(\hat{\Delta}, \tilde{\Delta})$	$(\hat{\Delta}, \bar{\Delta})$
	(a) Uniform Distribution			
0	0.28	0.32	1.12	1.16
0.2	0.16	0.26	1.62	0.95
0.3	0.01	0.31	346.8	1.01
	(b) Normal Distribution			
0	0.42	0.48	1.14	1.12
0.2	0.38	0.44	1.16	1.02
0.3	0.22	0.41	1.83	0.95
	(c) Double Exponential Distribution			
0	0.58	0.88	1.52	1.20
0.2	0.45	0.53	1.17	0.99
0.3	0.06	0.63	11.06	0.85
	(d) Cauchy Distribution			
0	209.7	400.8	1.91	1.27
0.2	622.8	761.7	1.22	1.10
0.3	115.8	579.6	5.01	0.97

Table 1 contains the empirical means and variances (appeared in parantheses) of the estimators in 500 replications. The empirical variance is computed as a sample variance of 500 estimates. The four estimators discussed in this paper are all biased. They actually overestimates the true value. For example, when $m=n=10$ and the distribution of Y is standard normal, the expected value of $\sqrt{F}$ is approximately 2.13 instead of 2. In this case the empirical mean of $\sqrt{F}$ was 2.12 and the standard deviation of the empirical mean in samples of size 500 is about 0.03. The relative efficiency of two estimators are computed as an inverse ratio of empirical variances. The results of the empirical relative efficiencies are presented in Table 2. The notation $(\hat{\theta}_1, \hat{\theta}_2)$ in Table 2 means the relative efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

From Table 1 and Table 2, we can see that $\hat{\Delta}$ is always superior to the Bhattacharyya estimator $\tilde{\Delta}$ if the distribution is symmetric. The nonparametric estimators are insensitive to the assumption of distributions. The parametric estimator $\sqrt{F}$ appears to be disastrous in Cauchy distribution.

Note that $\sqrt{F}$ and $\bar{\Delta}$ are computed from observations adjusted for sample means and sample medians, respectively, for each distribution, while $\tilde{\Delta}$ and $\hat{\Delta}$ are adjusted for the combined sample median. The results show that the proposed estimator $\hat{\Delta}$ is robust to the assumption of equal medians. But, the Bhattacharyya estimator $\tilde{\Delta}$ works poorly when the assumption of equal medians is significantly violated.

5. Concluding Remarks

In this paper we have proposed a class of estimators of the ratio of scale parameters $\Delta=\sigma_1/\sigma_2$, based on the Fligner-Killeen test. When the Wilcoxon scores are used, the explicit form of the estimator is obtained. A small-sample Monte Carlo study shows that the proposed estimator $\hat{\Delta}$ is, as expected, superior to the Bhattacharyya estimator $\tilde{\Delta}$. When the distributions are not symmetric, the ratio $|X-M|/|Y-M|$ of opposite signs does not provide any information about $\Delta=\sigma_1,/\sigma_2$. Thus, in this case the Bhattacharyya estimator is expected to be better than $\hat{\Delta}$ or $\bar{\Delta}$. The general behavior of the nonparametric estimators is much better than the parametric estimator in heavy-tailed Cauchy distribution. Table 1 also shows that the nonparametric estimators are more robust than the parametric estimator.

When the assumption of equal medians is violated, the Bhattacharyya estmator is too bad. It can be explained as follows. In the case with $\eta_1 \neq \eta_2$ the number of relevant pairs is getting smaller, and the ratios of relevant pairs are usually badly biased. In this case the estimator $\bar{\Delta}$, which is obtained from the observations adjusted for each sample median, is recommended. For moderate-tailed distributions the parametric estimator is, as expected, significantly better than any nonparametric estimators.

The asymptotic properties of $\hat{\Delta}$ and $\bar{\Delta}$ are not discussed in this paper. Fligner and Hettmansperger (1977) proved the asymptotic normality of complicated statistics using the fact that if the marginal distribution of the auxiliary statistic (e.g. sample median) converges strongly and the conditional distribution of the statistic of interest converges weakly then the marginal distribution of the statistic we are interested in converges weakly. Once the asymptotic normality of $h(V, W)$ in (2.3) is established, the asymptotic distributions of $\hat{\Delta}$ and $\bar{\Delta}$ can be obtained in a similar way to the derivation of asymptotic normality of $\tilde{\Delta}$ in Bhattacharyya (1977). We conjecture that $\hat{\Delta}$, $\bar{\Delta}$ and $\tilde{\Delta}$ are all asymptotically equivalent under the conditions in Section 1.

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