On the Srivastava's Theorem for the search design.

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ABSTRACT

In this paper, Srivastava's Theorem for the search design is considered, with additional assumptions, to the 3^n parallel flats fractions. It is also expressed in terms of ACPM.

1. Introduction

The basic mathematical formulation of the search design problem and some necessary and sufficient conditions for existence of designs were given by Srivastava (1975, 1976). In her first paper, "Designs for Searching Non-negligible Effects", she established the following definition and theorem which are basic in the search designs.

Definition 1.1. Consider the following general linear model

$$\underline{Y}=X_1\underline{\beta}_1+\underline{X}_2\underline{\beta}_2+\underline{e}, \qquad (1.1)$$
$$E(\underline{e})=\underline{0}, \quad V(\underline{e})=\sigma^2 I_N,$$

where $\underline{Y}(N\times 1)$ is a vector of observations, and $\underline{e}(N\times 1)$ is the error vector. The $\underline{X}_i(N\times\nu_i)$, $i=1,2$ are called design matrices, and $\underline{\beta}_i(\nu_i\times 1)$, $i=1,2$, are vectors of fixed unknown parameters with $\underline{\beta}_i'=(\beta_{i1},\beta_{i2}\ \beta_{i\nu_i})$. Suppose that we want to estimate the elements of $\underline{\beta}_1$. Also suppose it is given that the elements of $\underline{\beta}_2$ are all negligible, except possibly for a set of, at most, k elements where k is a known positive integer. The integer k would usually be quite small compared to ν_2. We want $\underline{Y}$ (and hence $\underline{X}_1$ and $\underline{X}_2$) to be such that we can estimate all elements of $\underline{\beta}_1$ and furthermore, search the

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non-negligible elements of $\underline{\beta}_2$ and estimate them. Let T be the design corresponding to the observations $\underline{Y}$. Then the design T is said to be a search design of resolving power k.

Theorem 1.1. Consider the model (1.1), together with $\underline{e}=\underline{0}$, Then T is a search design of resolving power k if, and only if, for every submatrix $X_{2k}(N\times 2k)$ of X_2 we have Rank $(X_1:X_{2k})=\nu_1+2k$.

In this paper, we consider a slight modification of the search problem for the 3^n factorial. Suppose that $\sigma^2=0$. Let $\underline{\beta}_1$ contain the general mean μ and the main effects in 3^n factorial so that $\nu_1=1+2n$. Let $\underline{\beta}_2$ denote the two-factor interactions so that $\nu_2=\binom{n}{2}\times 4$. We assume that, at most, k of the $\binom{n}{2}$ possible two-factor interactions are present. The three-factor and higher-order interactions are not present. In this situation, the design is said to be a search design of resolution III.k.

2. Some Basic Theorems

The parallel-flats fraction design T can be expressed as solutions to the symbolic matrix equation.

$A\underline{t}=C$ where $C=(\underline{c}_1\ \underline{c}_2 \ldots \underline{c}_f)$.

The parallel-flats fraction construction introduces the alias set structure on the columns of the X-matrix. This structure simplifies certain aspects of Theorem 1.1.

By virtue of the alias structure X can be expressed as $X=(X_{.0} \vdots X_{.1} \vdots X_{.2} \vdots \cdots X_{.u})$ where u is a number of alias sets and we can consider alias sets separately since columns in different sets are orthogonal.

It will be assumed that at most k of the $\binom{n}{2}$ possible two-factor interactions will be present. Since each two-factor interaction represents four degrees of freedom, each selection of k interactions corresponds to $4k$ columns of X. Similarly, each possible combination of $2k$ columns in Theorem 1.1 corresponds to $2k$ interactions here and thus sets of $8k$

columns. For example, if $k=2$ then for every set of four two-factor interactions we must consider 16 colmns and check $\binom{m}{4}$ cases where $m=\binom{n}{2}$.

Also the matrix $(X_1 : X_{2k})$ in Theorem 1.1 can be partitioned by alias sets here as $[X_{.01} : X_{.0,2k}) : (X_{.11} : X_{.1,2k}) : \cdots : (X_{.u1} : X_{.u,2k})]$ where $X_{.j1}$ corresponds to columns for linear and quadratic effects of main effects in the jth alias set and $X_{.j,2k}$ corresponds to columns for linear and quadratic effects of those interactions that occur in the jth alias set. This can be expressed as $[(P_{1m} : P_{1,2k}) : (P_{2m} : P_{2,2k}) : \cdots : (P_{um} : P_{u,2k})]$ in terms of ACPM where P_{jm} corresponds to columns for main effects and $P_{j,2k}$ corresponds to columns for effects of those interactions that occur in the jth ACPM. Note that rank $(X_{.j1} : X_{.j,2k})=2\text{rank}(P_{jm} : P_{j,2k})$, $j=1, 2, \ldots, u$. Therefore, $(X_{.j1} : X_{.j,2k})$ is full rank if, and only if, $(P_{jm} : P_{j,2k})$ is full rank where $j=1, 2, \ldots, u$. (For ACPM, see the paper 6).

Suppose that we have a search design of Resolution III. k. Then if k of the two-factor interactions are present in the model we must be able to estimate them. Therefore, we have the following theorem using an obvious generalization of the above notation.

Theorem 2.1. A necessary condition that T be a Search Design of Resolution III. k is that for every selection of k two-factor interactions the matrix $(X_{.j1} : X_{.j,k})$ is full rank for $j=0, 1, 2, \ldots, u$.

Corollary. A necessary condition that T be a Search Design of Resolution III.k is that for every selection of k two-factor interactions the matrix $(P_{jm} : P_{j,k})$ is full rank for $j=1, 2, \ldots, u$, and the matrix $(X_{.01} : X_{.0,k})$ is full rank.

The conditions of Theorem 1.1 when applied to this case would require that for every subset of $2k$ interactions the matrices $[X_{.j1} : X_{.j,2k}]$, $j=0, 1, 2, \ldots, u$, be full rank. Since the $2k$ interactions may occur in several different alias sets, this may not be too difficult to check. However, we will impose one further assumption which drastically simplifies this condition. The four degrees of freedom for any interaction, say F_i with F_j, partition into two parts F_iF_j and $F_iF_j^2$, each with two degrees of freedom usually called linear and quadratic. We will assume that if F_i and F_j interact, then at least one of the effects (linear and quadratic) in each of F_iF_j and $F_iF_j^2$ is nonzero.

This assumption is not as restrictive as it might appear. Suppose F_i and F_j are both quantitative variables and a polynomial in x_i and x_j is being fitted. If only one of the four terms x_ix_j, $x_i^2x_j$, $x_ix_j^2$, or $x_i^2x_j^2$ is required in the model (for example, only x_ix_j), then every one of the four degrees of freedom defined by F_iF_j and $F_iF_j^2$ linear and quadratic will be nonzero. The only way for these to be zero is for several of the terms to be required in the model and "accidently add to zero" in the linear combinations defining F_iF_j and $F_iF_j^2$. The probability that this occurs for both linear and quadratic in a set will be considered zero. Similarly, if one or both of F_i and F_j are qualitative it seems equally unlikely that these particular contrasts will be zero, hence that probability will be assumed to be zero.

We will examine this case in depth for $k=2$. Suppose that effects F_iF_j and $F_iF_j^2$ are in different alias sets and Theorem 2.1 is satisfied. Since $k=2$ we have to handle all possible subsets of four two-factor interactions. Consider F_iF_j, F_kF_l, $F_i'F_j'$, and $F_k'F_l'$ where $(i, j)\neq(k, l)\neq(i, j')\neq(k', l')$ such that none of the effects occur in S_0. Suppose the $8=4\times2$ effects appear in three alias sets such that two of the alias sets have three effects each and the third has two. This situation will be denoted by $(3, 3, 2, 0, 0, \dots)$. Similarly, if all eight components occur in two alias sets with four in each we will denote this by $(4, 4, 0, 0, \dots)$. Thus a vector $(a_1, a_2, \dots)$ denotes the case where a_1 effects in one alias set, a_2 in another, and so on. With this additional assumption we will establish the following theorem.

Theorem 2.2. Suppose that two effects F_iF_j and $F_iF_j^2$ are in different alias sets and for every selection of two two-factor interactions the matrix

$(X_{.j1}: X_{.j,2})$ is full rank, $j=0, 1, 2, \ldots, u$. Suppose also that if X_i and F_j interact that at least one degree of freedom from both F_iF_j and $F_iF_j^2$ are nonzero. Then a necessary and sufficient condition for a parallel-flats fraction T to be a Search Design of Resolution III.2 is that for any subset of four two-factor interactions:

(Case 1) If the division is $(4, 4, 0, 0, \ldots)$ then at least one of matrices $(X_{.r1} : X_{.r,4})$or$(X_{.q1} : X_{.q,4})$ is full rank wher r and q are indices of alias sets which contain four two-factor interactions.

(Case 2) If the division is $(4, 3, 1, 0, 0, \ldots)$ then at least one of matrices $(X_{.r1} : X_{.r,4})$or$(X_{.q1} : X_{.q,3})$ is full rank where r and q are indices of alias sets which contain four two-factor interactions and three two-factor interactions respectively.

(Case 3) Every other division is automatically resolvable.

Proof. For a parallel-flats fraction the columns of X are partitioned by alias sets and each alias set can be treated separately. This fact is perhaps most clearly seen from equation

$$E(\underline{L}_j)=H_j\underline{F}_j, j=0, 1, 2, \ldots, u, \text{ (See the paper6).}$$

where H_j depends only on $X_{.j}$. With $\sigma^2=0$ we have that $\hat{\underline{L}}_j=H_j\underline{F}_j$, $j=0, 1, 2, \ldots, u$, must be satisfied for every j, and these may be treated separately.

For the necessary portion Cases 1 and 2 suppose that both $[X_{.v1} : X_{.v,2k}]$, $v=1, 2$ are less than full rank. Then along the lines of Srivastava (1975) in th proof of Theorem 1.1 we can find two different models which exactly fit the data.

For the sufficiency in these two cases, we note that if one of the two is full rank we can estimate all effects and hence observe at least two to be zero. Then using the second assumption of the theorem we can "infer" that the whole interaction is negligible and eliminate these from the model. To see that all other cases are resolvable, note that at least two different effects will always be estimate. If both are zero, those interactions can be eliminated from the model. If one is zero and the other is nonzero, one factor can be eliminated and one necessarily retained which also resolves the detection. If both are nonzero we have, of course, completed the detection. This completes the proof.

Corollary. Suppose that two effects F_iF_j and $F_iF_j^2$ are in different alias sets and for every selection of two-factor interactions the matrix $(P_{jm} : P_{j,2})$ is full rank for $j=1, 2, \ldots, u$. Suppose also that if F_i and F_j interact that at least one degree of freedom from both F_iF_j and $F_iF_j^2$ are nonzero. Then a necessary and sufficient condition for a parallel-flats fraction T to be a Search Design of Resolution III.2 is that for any subset of four two-factor interactions:

(Case 1) if the division is $(4, 4, 0, 0, \ldots)$ then at least one of matrices $(P_{rm} : P_{r,4})$ or $(P_{qm} : P_{q,4})$ is full rank;

(Case 2) if the division is $(4, 3, 1, 0, 0, \ldots)$ then at least one of matrices $(P_{rm} : P_{r,4})$ or $(P_{qm} : P_{q,3})$ is full rank;

(Case 3) every other division is automatically resolvable.

3. Example

Consider the 3^6 factorial design with the design matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ The following alias sets are obtained.

$S_0 = \{\mu, F_3F_6^2, F_4F_5^2\}$

$S_1 = \{F_1, F_2F_3, F_2F_4^2, F_2F_5^2, F_2F_6, F_3F_4, F_3F_5, F_4F_6, F_5F_6\}$

$S_2 = \{F_2, F_1F_3, F_1F_4, F_1F_5, F_1F_6, F_3F_4^2, F_3F_5^2, F_4F_6^2, F_5F_6^2\}$

$S_3 = \{F_3, F_6, F_1F_2, F_1F_4^2, F_1F_5^2, F_2F_4, F_2F_5, F_3F_6\}$

$S_4 = \{F_4, F_5, F_1F_2^2, F_1F_3^2, F_1F_6^2, F_2F_3^2, F_2F_6^2, F_4F_5\}$.

The following ACPM are obtained with the five parallel flats

$$C=\begin{pmatrix} 0 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

$$\begin{array}{ccccccccc} F_1 & F_2F_3 & F_2F_4^2 & F_2F_5^2 & F_2F_6 & F_3F_4 & F_3F_5 & F_4F_6 & F_5F_6 \end{array}$$

$$P_1=\begin{pmatrix} e & e & e & e & e & e & e & e & e \\ e & (021) & (021) & (012) & e & (021) & e & (012) & (021) \\ e & (012) & (021) & e & (021) & e & (021) & (021) & (021) \\ e & e & (012) & (021) & (021) & (021) & (012) & e & (021) \\ e & (012) & e & (021) & (012) & (021) & e & (021) & e \end{pmatrix}$$

$$\begin{array}{ccccccccc} F_2 & F_1F_3 & F_1F_4 & F_1F_5 & F_1F_6 & F_3F_4^2 & F_3F_5^2 & F_4F_6^2 & F_5F_6^2 \end{array}$$

$$P_2=\begin{pmatrix} e & e & e & e & e & e & e & e & e \\ e & (021) & (012) & (021) & e & e & (021) & (021) & (012) \\ e & (012) & (012) & e & (021) & (012) & (021) & e & (012) \\ e & e & (021) & (012) & (021) & (012) & (021) & (021) & e \\ e & (012) & e & (012) & (012) & (021) & (012) & (021) & (012) \end{pmatrix}$$

$$\begin{array}{cccccccc} F_3 & F_6 & F_1F_2 & F_1F_4^2 & F_1F_5^2 & F_2F_4 & F_2F_5 & F_3F_6 \end{array}$$

$$P_3=\begin{pmatrix} e & e & e & e & e & e & e & e \\ e & (021) & (021) & (012) & e & e & (012) & (012) \\ e & (021) & (012) & e & (012) & (021) & (012) & (012) \\ e & (012) & e & (012) & (021) & (021) & (012) & (021) \\ e & e & (012) & (012) & e & (012) & (021) & e \end{pmatrix}$$

$$\begin{array}{cccccccc} F_4 & F_5 & F_1F_2^2 & F_1F_3^2 & F_1F_6^2 & F_2F_3^2 & F_2F_6^2 & F_4F_5 \end{array}$$

$$P_4=\begin{pmatrix} e & e & e & e & e & e & e & e \\ e & (012) & (021) & (012) & (021) & e & (021) & (021) \\ e & (021) & (021) & e & (012) & (012) & e & (012) \\ e & (021) & (012) & (012) & e & (012) & (021) & (012) \\ e & (012) & e & (012) & (012) & (021) & (021) & (021) \end{pmatrix}$$

Nate that effects F_iF_j and $F_iF_j^2$ are in different alias sets where $i \neq j \in \{1, 2, \ldots, 6\}$ and for every selection of two two-factor effects $(P_{jm} : P_{j,2})$ is full rank for $j=1, 2, 3, 4$. It is clear that $(X_{.01} : X_{.0,2})$ is full rank. Consider that the division is $(4, 4, 0, 0)$. There is only one case:

$\{F_3F_4,\ F_3F_5,\ F_4F_6,\ F_5F_6\}\ \varepsilon\ S_1$

$\{F_3F_4^2,\ F_3F_5^2,\ F_4F_6^2,\ F_5F_6^2\}\ \varepsilon\ S_2.$

From P_1, we get the following matrix.

$$\begin{pmatrix} e & e & e & e & e \\ e & (021) & e & (012) & (021) \\ e & e & (021) & (021) & (012) \\ e & (021) & (012) & e & (021) \\ e & (021) & e & (021) & e \end{pmatrix}$$

From P_2, we obtain

$$\begin{pmatrix} e & e & e & e & e \\ e & (021) & e & (021) & (012) \\ e & (012) & (021) & e & (012) \\ e & (012) & (021) & (021) & e \\ e & (021) & (012) & (021) & (012) \end{pmatrix}$$

It can be checked that these two matrices are full rank.

Next consider that the division is $(4, 3, 1, 0)$. There are $\binom{4}{3}\times 4\times 2=32$ cases which we must check. Let us consider one case.

$\{F_2F_3,\ F_3F_4,\ F_3F_5,\ F_4F_6\}\ \varepsilon\ S_1$

$\{F_3F_4^2,\ F_3F_5^2,\ F_4F_6^2\}\ \varepsilon\ S_2$

$\{F_2F_3\}\ \varepsilon\ S_4.$

From ACPM P_1 and P_2 the following matrices, which are full rank, are obtained respectively.

$$\begin{pmatrix} e & e & e & e & e \\ e & (021) & (021) & e & (012) \\ e & (012) & e & (021) & (021) \\ e & e & (021) & (012) & e \\ e & (012) & (021) & e & (021) \end{pmatrix} \qquad \begin{pmatrix} e & e & e & e \\ e & e & (021) & (021) \\ e & (012) & (021) & e \\ e & (012) & (021) & (021) \\ e & (021) & (012) & (021) \end{pmatrix}$$

Similarly, we can show that at least one of the matrices is full rank for each case. Hence the parallel flats design T satisfies the corollary of Theorem 2. 2.

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