Alias Component Permutation Matrices (ACPM) for the 3^n Parallel Fractional Factorial Design

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ABSTRACT

A parallel flats faraction for the 3^n factorial experiment is symbolically written as $At=C(r\times f)$ where $A(r\times n)$ is of rank r. The A-matrix partitions the non-negligible effects into $(3^{n-r}-1)/2+1$ alias sets. The U_i effects in the i-th alias set are related pairwise by elements from S_3, the symmetric group on three symbols. For each alias set the f flats produce an $f\times u_i$ alias componet permutation matrices (ACPM) with elements from S_3. All the the information concering the relationships among levels of the effects is contained in the ACPM.

1. Introduction

In this paper, a treatment combination or assembly for the 3^n experiment is denoted by $n\times 1$ vector t with elements from $\{0, 1, 2\}$ indicating the levels of the n factors. A fractorial-factorial design is some collection of N treatment combinations, denoted by T. Specifically, the symbol T is used to denote either the set of treatment combinations in the design or the $n\times N$ matrix having columns $t_1, t_2, \ldots, t_N$. In the experiment observations are made at each treatment combination in T. The observation correponding to t is denoted by y_t, and the vector of all N observations is denoted by $N\times 1$ vector y.

For convenience of notation, let $T'=D_M=[d_1 d_2 \ldots d_n]$, so D_M (M referring to main-effect design matrix) is an $N\times n$ matrix with rows $t'_1, t'_2, \ldots, t'_N$. For $i=1, 2, \ldots, n$, the column d_i of D_M contains the levels for factor F_i. If for any i and j an interaction between factor F_i and factor F_j is to be included in the model, the two columns

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$$d_i+d_j \text{ and } d_i+2d_j, \ \bmod(3)$$

must be adjoined to D_M; and in the usual notation the corresponding interaction effects are denoted by $F_iF_j^x$, $x \neq 0 \in \{1, 2\}$.

This procedure is followed for every pair of factors that interact, and D represents the design matrix that results after all appropriate columns corresponding to interactions have been adjoined to D_M. For example, if all pairs of factors interact, there will be $2\binom{n}{2}$ columns adjoined to D_M to form the matrix D. The defining vector e for effect $F_iF_j^x$ will have ith component 1, jth component x, and the remaining zero. In general, the defining vector for an effect $F_{i_1}^{x_1}F_{i_2}^{x_2}\ldots F_{i_k}^{x_k}$ will have x_j in the i_j th position and zeros elsewhere.

Since each factor appears at three levels, each main effect symbol, F_i, $i=1, \ldots, n$, denotes two degrees of freedom. Similarly, if factors F_i and F_j interact, each of the two symbols $F_iF_j^x$, $x=1, 2$, denotes two degrees of freedom, thus accounting for all $2^2=4$ degrees of freedom associated with the interaction. These single degrees of freedom are defined by the set of orthogonal contrasts in Table 1. 1.

TABLE 1. 1. Coefficients of Three Level Orthogonal Contrasts

Level	L	Q
0	−1	1
1	0	−2
2	1	1

Using the usual notation, we write the model equations $E(y)=XF$. The parameter vector F includes the general mean μ along with two single degree of freedom components for each main effect and four single degree of freedom components for each two-factor interaction. The first column of the design matrix X is a colum with every element one, corresponding to μ. The remainder of X is formed by replacing each element of D by the corresponding row of Table 1. 1. Questions of estimability relate to the rank of X and linear dependencies among the columns of X.

The regular 3^{n-r} fractions of the 3^n factorial are solutions mod(3) to

$$At=C, \tag{1. 1}$$

Where A is an $r\times n$ matrix of rank r and c is $r\times 1$, both over{0, 1, 2, } mod (3) It is an easy task to write the alias sets from Equation(1. 1). Geometrically, we may regard the 3^n treatment combinations as points in the Euclidean geometry of order n over the field of order 3, denoted by $EG(n, 3)$. In this interpretation Equation (1. 1) represents a linear $(n-r)$-flat of 3^{n-r} points in $EG(n, 3)$.

An alternative to choosing points on a single flat in $EG(n, 3)$ is to consider the union of points on several flats. Thus consider the flats generated by equations

$$A_i t=C_i, \qquad i=1, 2, \ldots, f, \tag{1. 2}$$

where A_i is $r_i\times n$ of rank r_i and c_i is $r_i\times 1$. The design T corresponding to (1. 2) is

$$T=\bigcup_{i=1}^{f}\{t:A_i t=c_i\} \tag{1. 3}$$

The ith flat contains 3^{n-r_i} points, but since the flats may intersect in various ways the number of points in T as well as the estimability of factorial effects depend on the A_i and C_i in a rather complex manner. Consideration of designs of type (1. 3) was motivated by a search for a general series of minimal or near minimal resolution IV designs for the s^n factorial, Anderson and Thomas (1977, 1979).

A special case of (1. 3) is when $A_1=A_2=\ldots=A_f$, each of rank r. In this case the linear flats are parallel, and the fraction contains $f\times 3^{n-r}$ treatment combinations assuming $c_i\neq c_j$, $i\neq j=1, 2, \ldots, f$. Such a fraction will be termed a parallel flats fraction. The treatment combinations in a parallel flats fraction will be symbolically expressed by the matrix equation

$$At=C, \text{ where } C=[c_1,c_2,\ldots,c_f].$$

A 3^n parallel flats fraction T consists of assemblies obtained as solutions to the symbolic matrix equation $At=C$ where without loss of generality consider

A to be in the canonical form

$$A=[A^*;Ir].$$

Let $R(A)$ represent the row space of A and let e be the defining vector of effect E. Suppose $e' \ni R(A)$. Then there exists a vector v such that $v'A=e'$. Let T_i consist of the 3^{n-r} assemblies obtained as solutions to $At=c_i$. For each assembly t^* in T_i, $e't^*=v'c_i=c^*$; that is, only the single value $c^* \in \{0, 1, 2\}$.

Let $X_{i\cdot}$ represent the rows of the X-matrix corresponding to T_i. The two columns of $X_{i\cdot}$ corresponding to the linear and quadratic components of E are obtained from Table 1, 1 depending on c^*. The rank of the columns of $X_{i\cdot}$ corresponding to the mean and to the linear and quadratic components of E is one. This is true for all columns corresponding to any effects with defining vectors in $R(A)$. The estimate of μ from T_i alone is an estimate of a linear combination involving the mean, linear and quadratic components of any effects which have defining vectors in $R(A)$. The linear combination estimated depends on the flat chosen and the effects found in S_0. The running of the treatment combinations in a particular flat confounds the effects found in the alias set containing the mean which merely means the individual effects are not separable. This feature is the basis for saying that effects consitute an alias set S_0.

The weight of a vector v denoted by $w(v)$, is defined to be the number of nonzero elements of v. If three-factor and higher-order interactions are suppressed, then all e' $R(A)$ such that $w(e) \leq 2$ define those effects which are aliased with the mean. It may happen that S_0 consists of only μ.

Definition 1. 1. Let e' define the effect E. Effect E is aliased with the mean, if and only if,

$$\text{rank}\begin{bmatrix} A \\ \cdots\cdots \\ e' \end{bmatrix}=\text{rank}(A)=r.$$

Let E be an effect that is not aliased with the mean. The defining vector of E is added to all row vectors in $R(A)$ to establish the aliasing structure with respect to E. The resulting row vectors define those effects which are aliased with D.

Definition 1.2. Let e_1 and e_2 be the defining vectors of effect E_1 and effect E_2 and suppose

$$\text{rank}\begin{bmatrix} A \\ \cdots\cdots \\ e_1{}' \end{bmatrix} = \text{rank}\begin{bmatrix} A \\ \cdots\cdots \\ e_2{}' \end{bmatrix} = r+1.$$

Then E_1 is aliased with E_2, if and only if,

$$\text{rank}\begin{pmatrix} A \\ \cdots\cdots \\ e_1{}' \\ \cdots\cdots \\ e_2{}' \end{pmatrix} = r+1.$$

The effects are thus partitioned into sets called alias sets.

Aliasing is an equivalence relation. The effects of F are partitioned into alias sets S_0, $S_1, \ldots, S_u$ which are determined solely by A. The number of alias sets not containing the mean is $(3^{n-r}-1)/2$; however, some of these may by empty. Thus $u \leq (3^{n-r}-1)/2$ and the numberof effects in S_j will be denoted by u_j.

If the ordering defined by F is imposed on the columns of the X-matrix resulting from a 3^n parallel flats fraction T constructed from f flats then

$$X = \begin{pmatrix} X_{10} & X_{11} & \cdots & X_{1u} \\ X_{20} & X_{21} & \cdots & X_{2u} \\ \vdots & \vdots & & \vdots \\ X_{f0} & X_{f1} & \cdots & X_{fu} \end{pmatrix}$$

where X_{ij}, i $1, 2, \ldots, f$; $J \in \{0, 1, \ldots u\}$, represents the partition of the X-matrix with rows corresponding to the ith flat and columns corresponding to the effects in the jth alias set. In addition, $X_{i\cdot}$ is used to represent the partition of the X-matrix with rows corresponding to the ith flat and $X_{\cdot j}$ is used to represent the partition of the X-matrix with columns corresponding to the jth alias set. Therefore,

$$X = \begin{pmatrix} X_{1\cdot} \\ \cdots\cdots\cdots \\ X_{2\cdot} \\ \cdots\cdots\cdots \\ \vdots \\ \cdots\cdots\cdots \\ X_{f\cdot} \end{pmatrix} = \begin{bmatrix} X_{\cdot 0} \,\vdots\, X_{\cdot 1} \,\vdots\, \cdots \,\vdots\, X_{\cdot u} \end{bmatrix}$$

It is easy to show that rank $(X_{i0})=1$, rank$(X_{ji})=2$, $J=1, 2, \ldots., u$, and rank $(X_{i\cdot})=1+2u$.

2. Alias Component Permutation Matrices

The components of F will be ordered so that the first component is and the next positions will be occupied by the linerar and quadratic components of effects aliased with the mean. The vector of these effects will be denoted by F_0. The next positions of F will be filled with the linear and quadratic components of effects in the first alias set and will be denoted by F_1. The remaining positions in F are filled alias sets at a time and denoted by F_2, $F_3, \ldots, F_u$. $F=(F_0' F_1' F_2' \ldots F'_u)'$. Therefore, we have $y_i=X_{i\cdot}F=X_{i0}F_0+X_{i1}F_1+X_{i2}F_2+\ldots+X_{iu}F_u$, where F_j denotes the vector of effects in the jth alias set, $j=0, 1, \ldots u$.

For the ith flat, let S_{ij} and $S_{ij}{}^2$ denote linear and quadratic effects identified with the jth alias set, $j=0, 1, \ldots, u$. Typically some effects in the jth alias set will be taken to represent the set and hence that effect will be identified as S_{ij}. The single degree of freedom effect S_{i0} will always be identified with. Then for every j we can choose two columns from X_{ij} corresponding to the effect S_{ij}, $j=0, 1, \ldots u$ and the vector 1 corresponding to the effect. Let these $1+2u$ columns form a matrix Z_i, with the first column 1.

Consider the relation $y_i=Z_iS_i$, where $S_i'=(S_{i0}, S_{i1}, S_{i1}{}^2, \ldots, S_{iu}, S_{iu}{}^2)$. Then $S_i=(Z_i'Z_i)^{-1}Z_i'y_i$, where $S_i'=(S_{i0}, S_{i1}, S_{i1}{}^2, \ldots, S_{iu}, S_{iu}{}^2)$, is the vector of estimates of S_i for the ith flat. Since $y_i=X_iF$, we have $S_i=(Z_i'Z_i)^{-1}Z'_iy_i=(Z_i'Z_i)^{-1}Z_i'X_i$, $F=M_iF$, so that the elements of S_i are linear combinations of the elements of F. It is clear that

$$
\begin{array}{c}
\\
\text{Mi}=
\end{array}
\begin{array}{c}
\begin{array}{cccc}
F_0 & F_1 & \cdots & F_u
\end{array}\\
\left(\begin{array}{c:c:c:c}
\text{one nonzero row} & 0 & 0 & 0 \\ \hdashline
0 & \text{two nonzero rows} & 0 & 0 \\ \hdashline
0 & 0 & \ddots & 0 \\ \hdashline
0 & 0 & 0 & \text{two nonzero rows}
\end{array}\right)
\end{array}
\qquad (2.1)
$$

Now consider f flats. For each i the matrix M_i is of the form (2. 1) but the elements of two nonzero rows are different from each other, $i=1, 2, \ldots, f$. For every j we have f estimates of S_{ij} identified with the jth alias set, $j=0, 1, 2\ldots, u$. Let

$$L_0=\begin{pmatrix} S_{10} \\ S_{20} \\ \vdots \\ S_{f0} \end{pmatrix}, \quad L_1=\begin{pmatrix} S_{11} \\ S^2{}_{11} \\ S_{21} \\ S^2{}_{21} \\ \vdots \\ S_{f1} \\ S^2{}_{f1} \end{pmatrix}, \quad L_j=\begin{pmatrix} S_{1j} \\ S^2{}_{1j} \\ S_{2j} \\ S^2{}_{2j} \\ \vdots \\ S_{fj} \\ S^2{}_{fj} \end{pmatrix}, \ldots, \quad L_u=\begin{pmatrix} S_{1u} \\ S^2{}_{1u} \\ S_{2u} \\ S^2{}_{2u} \\ \vdots \\ S_{fu} \\ S^2{}_{fu} \end{pmatrix}.$$

Suppose that $F_i'=(E_1, E_2, \ldots, E_{u_i})$. Then it is clear that

$$E(L_j)=H_jF_j, \qquad j=1, 2, \ldots, u,$$

$$\text{where } H_j=\begin{pmatrix} \text{two nonzero rows in column } F_j \text{ of } M_1 \\ \text{two nonzero rows in column } F_j \text{ of } M_2 \\ \cdots \\ \text{two nonzero rows in column } F_j \text{ of } M_f \end{pmatrix}, \quad F_j=\begin{pmatrix} E_1 \\ E_1{}^2 \\ \vdots \\ Eu_j \\ Eu^2{}_j \end{pmatrix}.$$

Suppose that the effect S_{ij} has been identified as E_1, Then in any flat the levels of each other effect in the alias set are related to the levels of S_{ij} by one of the permutations in the symmetric group on three symbols {e, (012), (021), (01), (02), (12)}. For example, suppose the levels are related by (012) as in the array $\begin{bmatrix} 0 & 1 \\ 2 & 1 \\ 2 & 0 \end{bmatrix}$. The corresponding X-matrix is $\begin{bmatrix} -1 & 1 & 0 & -2 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$ and in estimating S_{ij} we have two linear combinations

$$\begin{pmatrix} 2 & 0 & -1 & 3 \\ 0 & 6 & -3 & -3 \end{pmatrix}, \text{ that is, } \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Therefore, the effect is related to S_{ij} by

$$\begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Let

$$D_e=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ D_{(012)}=\begin{pmatrix}-\frac{1}{2} & \frac{3}{2}\\ -\frac{1}{2} & -\frac{1}{2}\end{pmatrix}\ D_{(021)}=\begin{pmatrix}-\frac{1}{2} & -\frac{3}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix}$$
$$D_{(01)}=\begin{pmatrix}\frac{1}{2} & \frac{3}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{pmatrix},\ D_{(02)}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\ D_{(12)}=\begin{pmatrix}\frac{1}{2} & -\frac{3}{2}\\ -\frac{1}{2} & -\frac{1}{2}\end{pmatrix}. \tag{2.2}$$

Note that every effect in the set F_j is related to the effect S_{ij}, identified with the F_j by one of $\{D_e, D_{(012)}, D_{(021)}, D_{(01)}, D_{(02)}, D_{(12)}\}$. Suppose that the effects in the set F_j are related to the effect S_{ij} by $g_1, g_2, \ldots, g_{u_j}$, respectively, where $g\in\{e, (012), (021), (01), (02), (12)\}. =1, 2, \ldots, u_j$. Then E

$$\begin{bmatrix}\hat{S}_{ij}\\ \hat{S}^2_{ij}\end{bmatrix}=[Dg_1\ Dg_2 \ldots Dg_{u_j}]\begin{pmatrix}E_1\\ E_1{}^2\\ \vdots\\ Eu_j\\ E^2{}_{uj}\end{pmatrix}.$$

That is, every two nonzero rows is composed of six 2×2 matrices in (2.2). Hence we have

$$H_j=\begin{pmatrix}Dg_{11} & Dg_{12} & \cdots & Dg_{1u_j}\\ Dg_{21} & Dg_{22} & \cdots & Dg_{2u_j}\\ \vdots & \vdots & & \vdots\\ Dg_{f1} & Dg_{f2} & & Dg_{fu_j}\end{pmatrix}$$

where f is the number of flats and u is the number of effects in the jth alias set.

One important simplification of the above arguments is through the group of permutations of three symbols $\{0, 1, 2\}$, $H=\{e, (012), (021), (01), (02), (12)\}$ with the right multiplication.

For each alias set, we have defined matrix H_j and the elements of H_j are compose of D_e, $D_{(012)}, D_{(021)}, D_{(01)}, D_{(02)}$, and $D_{(12)}$. From the H_j we make the matrix P_j by replacing Dg by g. This matrix is called the alias component permutation matrix (ACPM).

For each alias set, choosing the first element in the set as the identified effect, it is clear that the elements of the first column of ACPM are e. If the first column of C is 0 then the first row of ACPM consists of e or the elements of {(01), (02), (12)}. In order to make all the elements of the first row e, postmultiply P_j by $D_j=\text{diag}(g_{11}^{-1}, g_{12}^{-1}, \ldots, g_{1u_j}^{-1})$, where g_{11}^{-1} denotes the inverse of the element in the first row and ith column. Then it can be shown that all the elements of P_j come from the subgroup $J_1=\{e,$ (012), (021)}.

Example 2.1. Consider a 3^4 factorial experiment for which it can be assumed that all three-and four-factor interaction effects are negligible. It is also assumed that among the $\binom{4}{2}=6$ possible pairs of two-factor interactions, at most two are nonzero. The A-matrix for this example will be taken as

$$A=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix},$$

thus there are flats of size nine. The alias sets are

$$S_0=\{\mu\},\quad S_1=\{F_1, F_2F_3, F_2F_4^2, F_3F_4\},\quad S_2=\{F_2, F_1F_3, F_1F_4, F_3F_4^2\},$$
$$S_3=\{F_3, F_1F_2, F_1F_4^2, F_2F_4\},\quad S_4=\{F_4, F_1F_2^2, F_1F_3^2, F_2F_3^2\}.$$

An example of a parallel-flats fraction in 27 runs is given with $C=(c_1c_2c_3)$ as $C=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$. By choosing the main effect in each alias set as the identified effect, the alias component permutation matrices are

$$P_1=\begin{array}{c} \begin{matrix} F_1 & F_2F_3 & F_2F_4^2 & F_3F_4 \end{matrix} \\ \begin{bmatrix} e & e & e & e \\ e & e & (021) & (012) \\ e & (021) & (012) & e \end{bmatrix} \end{array};\quad P_2=\begin{array}{c} \begin{matrix} F_2 & F_1F_3 & F_1F_4 & F_3F_4^2 \end{matrix} \\ \begin{bmatrix} e & e & e & e \\ e & e & (012) & (021) \\ e & (021) & (021) & (021) \end{bmatrix} \end{array}$$

$$P_3=\begin{array}{c} \begin{matrix} F_3 & F_1F_2 & F_1F_4^2 & F_2F_4 \end{matrix} \\ \begin{bmatrix} e & e & e & e \\ e & e & (021) & (012) \\ e & (021) & e & (012) \end{bmatrix} \end{array};\quad P_4=\begin{array}{c} \begin{matrix} F_4 & F_1F_2^2 & F_1F_3^2 & F_2F_3^2 \end{matrix} \\ \begin{bmatrix} e & e & e & e \\ e & (021) & (021) & (021) \\ e & (012) & e & (021) \end{bmatrix} \end{array}$$

3. Easymethod to obtain ACPM

Fortunately, it is easy to obtain the ACPM matrices directly from the equation $At=C$. Consider a single flat corresponding to $c=(c_1, c_2, \ldots, c_r)$,

$$[A^* : I_r]\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}=c.$$

The 3^{n-r} solutions to this equation are of the form

$$\begin{bmatrix} t_1 \\ c-A_* t_1 \end{bmatrix}$$

where t_1 assumes all possible 3^{n-r} possible values. Thus the points on the ith flat are expressed in terms of c.

Let $(e_1, e_2, \ldots, e_{u_j})$ be the defining vectors for the effects in an alias set where again e_1 corresponds to the defining effect S. Consider the product

$$\begin{bmatrix} t_1 \\ c-A^* t_1 \end{bmatrix}(e_1, e_2, \ldots, e_{u_j}).$$

From the 3^{n-r} possible such products choose three for which $\begin{bmatrix} t_1 \\ c-A^* t_1 \end{bmatrix} e_1 =$ 0, 1, and 2 respectively. Then using thesethree, say t_0, t_1 and t_2, compute the product $\begin{bmatrix} t_0' \\ t_1' \\ t_2' \end{bmatrix}[e_1, e_2, \ldots, e_{u_j}]$. The last u_j-1 columns of this matrix will involve c and implicitly define the permutation relationship of each element with E_1 in terms of c. Each column will be of the form

$$\begin{bmatrix} t_0' e_i \\ t_0' e_i + 1 \\ t_0' e_i + 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} t_0' e_i \\ t_0' e_i + 2 \\ t_0' e_i + 1 \end{bmatrix}$$

If it is of this second form, multiply the column by 2 so it becomes $\begin{bmatrix} 2t_0' e_i \\ 2t_0' e_i + 1 \\ 2t_0' e_i + 2 \end{bmatrix}$. The first row of the resultant marix is of the form

$$C^* = (0, x_2 t_0' e_2, x_3 t_0' e_3, \ldots, x_{u_j} t_0' e_{u_j}).$$

and the remaining rows are obtained by adding 1 and 2 respectively. The corresponding row of the ACPM is obtained from the first row by setting

$0 \to e$, $1 \to (012)$, and $2 \to (021)$.

The following example will illustrate.

Example 3. 1. Consider the 3^4 parallel flats in Example 2. 1 with A= $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$. The treatment combination on any flat are of the form

$$\begin{bmatrix} t_1 \\ t_2 \\ c_1 - t_1 - t_2 \\ c_2 - t_1 - 2t_2 \end{bmatrix}$$

where t_1 and t_2 take all values 0, 1, 2. The first alias set S_1 has defining vectors

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

and since the first effect depends only on the first component we can select the three combinations $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, Hence

$$\begin{bmatrix} 0 & 0 & c_1 & c_2 \\ 1 & 0 & c_1-1 & c_2-1 \\ 2 & 0 & c_1-2 & c_2-2 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix} = \begin{bmatrix} 0 & c_1 & 2c_2 & c_1+c_2 \\ 1 & c_1+2 & 2c_2+1 & c_1+c_2+1 \\ 2 & c_1+1 & 2c_2+2 & c_1+c_2+2 \end{bmatrix}.$$

The second column must be multiplied by 2 get in proper form, hence the first row is

$$C^* = [0 \;\; 2c_1 \;\; 2c_2 \;\; c_1+c_2].$$

For the parallel-flats fraction with $C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ the ACPM is thus

$$P_1 = \begin{bmatrix} e & e & e & e \\ e & e & (021) & (012) \\ e & (021) & (012) & e \end{bmatrix}.$$

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