論文80－17－6－7

Optimal Output P and PI Feedback for Discrete Time Systems
(離散시스템을 위한 最適出力 P & PI 궤환)

申 鉉 哲*, 卞 增 男**

(Shin, Hyun Chul and Bien, Zeungnam)

요 약

시간에 따라 변하지 않는 선형 다변수 시스템의 출력 궤환 제어를 이산시간계에서 최적화 하기 위한 필요 조건을 유도했다.

Quadratic performance index를 사용했다.

결과는 출력의 비례적분 궤환의 최적이득을 구하는 데에도 쉽게 적용할 수 있다. 예제를 들어 최적 궤환이득을 구했다.

Abstract

For linear discrete-time time-invariant multi-input multi-output systems, a necessary condition which an optimal output proportional feedback gains must satisfy is derived. Quadratic performance index is used. The result is extended to the design problem for determining optimal output proportional plus integral feedback gains. For illustration, an example problem is solved and discussed.

I. Introduction

As the digital devices of various kinds are readily available for use, the number of applications of the digital computer as a control device is also increasing. In fact, nearly all of the modern control algorithms may be implemented by digital devices, such as microcomputers, far more easily than by classical analog devices.

For digital control systems, time must be quantized and sampling is necessary. When the sampling period is extremely small compared with the dominant time constant of the closed loop system, we can approximate the discrete-time system as a continuous-time version for control without resulting in any serious degradation of performance. In many cases, however, it is impractical to make the sampling time very short, and the performance of the sampled data system depends on the sampling time. In some cases, even the controllability may be lost by sampling [12], in implementating digital controllers.

In this paper, the design of discrete-time output proportional and output proportional plus integral controllers is considered for linear time-invariant processes. It is well known [13] that the optimal control law for quadratic criterion can be obtained by employing feedback from all the states of the system. In practical applications, all the state variables are not always accessible. Even when all the states of a system are accessible, some difficulties still remain due to excessive instrumentation and cost requirements [11]. Several authors suggested alternatives in which only constrained states are used in feedback.

*準會員, 금오공대전자공학과

**正會員, 韓國科學院 電氣 및 電子工學科
(Dept. of Electrical Science, KAIS)

接受日字：1980年 6月 18日

Levine and Athans [1] derived algebraic necessary conditions to find optimal constant output feedback gains for continuous multivariable systems control, and computational considerations are given in [3], [4], [7], [8] and [10]. Recently, Hutcheson [9] obtained the same necessary conditions by direct derivation. In [6], Berger formulated a discrete-time output feedback problem and proposed an algorithm for finding the controller gain matrix.

In section II, a necessary condition which is the discrete version of the one in [1], is derived for optimal constant output feedback control with respect to a quadratic performance index. In section III, an extension is made to optimal output proportional plus integral feedback control. In this type of control, the steady state error will go to zero in the face of arbitrary large variations of uncertainties in the system parameters, provided that the closed loop system remains stable and the system is time-in-variant [11]. Finally an example problem is solved in section IV.

II. Optimal constant output proportional feedback.

Consider a linear time-invariant system whose n x 1 state vector x(k), m x 1 control vector u(k), and r x 1 output vector y(k) are related by

$$x(k+1) = A\,x(k) + B\,u(k) \qquad (2\text{-}1)$$

$$y(k) = C\,x(k) \qquad (2\text{-}2)$$

The performance measure is given as a standard quadratic form

$$J_0 = \tfrac{1}{2}\sum_{k=0}^{\infty} [\, x(k)'\,Q\,x(k) + u(k)'\,R\,u(k)\,] \qquad (2\text{-}3)$$

where Q is a n x n symmetric positive semidefinite matrix and R is a m x m symmetric positive definite matrix.

It is well known[13] that, if all the components of the state variables are abailable for feedback, equations (2-1) and (2-3) result an optimal control of the form

$$u(k) = -G\,x(k) \qquad (2\text{-}4)$$

where G is a real constant matrix with m rows and n columns.

Now consider the constraint that the control law u(k) be generated from output proportional feedback with time-invariant feedback gains. Then the control law becomes

$$u(k) = -F\,y(k) \qquad (2\text{-}5)$$

or

$$u(k) = -F\,C\,x(k) \qquad (2\text{-}6)$$

where F is a feedback matrix to be determined. By substituting equation (2-6) into (2-1), one can get

$$x(k+1) = (A - BFC)\,x(k), \qquad (2\text{-}7)$$

Furthermore

$$x(k) = (A - BFC)^k\,x(o). \qquad (2\text{-}8)$$

Using equations (2-6) and (2-8), equation (2-3) becomes

$$J_0 = \tfrac{1}{2}\,x(o)' \sum_{k=o}^{\infty} (A-BFC)'^k\,(Q+C'F'RFC) \cdot (A-BFC)^k\,x(o)$$

$$= \tfrac{1}{2}\,\mathrm{tr}[\, \sum_{k=o}^{\infty} (A-BFC)'^k\,(Q+C'F'RFC)\,(A-BFC)^k \cdot x(o)x(o)'\,] \qquad (2\text{-}9)$$

When the system matrices (A,B,C) are given, J_0 depends on F and x(0). By using inital state averaging, we may assume that $E\,[x(o)x(o)'] = X_o$. Here X_o can be used as a design parameter, if the system tends to be disturbed to some particular initial state. When no such information is available, a simple and frequently used way of eliminating the dependence on the inital state is to choose

$$X_o = I \text{ (identity)}. \qquad (2\text{-}10)$$

From now on in this section, the following form of J is used as a performance oriterion, if not stated differently.

$$J = \tfrac{1}{2}\,\mathrm{tr}\,[\, \sum_{k=o}^{\infty} (A - BFC)'^k\,(Q+C'F'RFC)\,(A - BFC)^k\,X_o\,]. \qquad (2\text{-}11)$$

Theorem II-1

Let F be an m x r constant matrix, and let

$$A_F = A - BFC. \qquad (2\text{-}12)$$

Assume that A_F is asymptotically stable. In order for F to be optimal with respect to equation (2-11), it is necessary that

$$RFCLC' - B'MA_FLC' = 0 \tag{2-13}$$

where L and M are respectively the unique symmetric positive semidefinite solutions of

$$A_F LA_F' - L = -X_o \tag{2-14}$$

$$A_F'MA_F - M = -(Q+C'F'RFC) \tag{2-15}$$

Proof:

Assume that there exists an F for which A_F = A-BFC is asymptotically stable. The necessary condition for minimizing J is that

$$\frac{\partial J}{\partial F} = 0 \tag{2-16}$$

Now let's show that equation (2-16) implies equation (2-13).

Note that, by choosing to the first order in ϵ,

$$[A - B(F + \epsilon\Delta F) C]^k \cong (A - BFC)^k$$

$$- \epsilon \sum_{i=o}^{k-1} (A - BFC)^{k-i-1} B\,\Delta FC(A-BFC)^i \tag{2-17}$$

Hence

$$J(F+\epsilon\Delta F) - J(F) \cong \tfrac{1}{2}\,\epsilon\, \mathrm{tr}[\sum_{k=o}^{\infty} 2\, CA_F{}^k X_o A_F{}'^k C'F'R\Delta F$$

$$- \sum_{k=1}^{\infty}\sum_{i=o}^{k-1} 2CA_F{}^i X_o A_F{}'^k (Q+C'F'RFC)\, A_F{}^{k-i-1} B\Delta F] \tag{2-18}$$

Using well-known Kleinman's lemma [1],

and using $\sum_{k=1}^{\infty}\sum_{i=o}^{k-1} f(i,k) = \sum_{i=o}^{\infty}\sum_{k=i+1}^{\infty} f(i,k)$, one obtains

$$\frac{\partial J}{\partial F} = RFC\,(\sum_{k=o}^{\infty} A_F{}^k X_o A_F{}'^k)\, C'$$

$$-\sum_{i=o}^{\infty}\sum_{k=i+1}^{\infty} B'A_F'{}^{k-i-1} (Q+C'F'RFC)\, A_F{}^k X_o\, A_F{}^i C' \tag{2-19}$$

By letting j = k–i–1 or k = i+j+1

$$\frac{\partial J}{\partial F} = RFCLC' - B'MA_FLC' \tag{2-20}$$

where

$$L = \sum_{k=o}^{\infty} A_F{}^k X_o A_F{}'^k \tag{2-21}$$

$$M = \sum_{k=o}^{\infty} A_F{}'^k (Q+C'F'RFC) A_F{}^k \tag{2-22}$$

Lemma II-2

If A is asymptotically stable, i.e. all the characteristic values of A have moduli strictly less than 1, then the discrete Lyapunov type equation (2-23) has a unique solution P.

$$A'PA - P = -Q \tag{2-23}$$

where A, P and Q are real n x n matrices with P and Q symmetric.

Lemma II-2 can be easily proved from [2].

From equations (2-21) and (2-22), we can see that L and M are at least positive semidefinite matrices. By choosing a positive definite X_o, we can guarantee the positive definiteness of L.

The equivalence of equations (2-14) and (2-15) and equations (2-21) and (2-22) can be verified by direct substitution of (2-21) and (2-22) into (2-14) and (2-15) respectively, and by showing the uniqueness of the solutions of the equations (2-14) and (2-15). The uniqueness of the solutions of these equations is shown in lemma II-2 given above. This completes the proof of theorem II-1.

Remark 1.

Throughout this derivation, it is assumed that a feedback matrix F which stabilizes A_F exists. For such F, J is finite and there exists an optimal F which minimizes the value of J. If no such F exists then the problem is meaningless.

Remark 2.

Equations (2-14) and (2-15) are in the form of well known discrete type Lyapunov equations. This conforms that A_F is an asymptotically stable matrix, if one of the following conditions is satisfied.

(i) For a positive definite X_o, there is a positive definite matrix L which satisfies equation (2-14).

(ii) For a positive definite matrix $Q+C'F'RFC$, there is a positive definite matrix M which satisfies equation (2-15).

From equations (2-11) and (2-22), one can see that

$$\min_F J(F) = ½ \, \mathrm{tr}\, [MX_o]$$

Remark 3.

Consider the relation between output feedback and state feedback. State feedback control may be considered as a special case of output feedback control such that C = I (identity)

With nonsigular matrix C, and positive definite matrix L, equation (2-13) gives.

$$F = R^{-1} B' M A_F C^{-1} \tag{2-24}$$

Using equation (2-12), one can get equation (2-25) from equation (2-24).

$$A_F = [I + BR^{-1} B'M]^{-1} A \tag{2-25}$$

Combing equations (2-15), (2-24), and (2-25), one can obtain the following matrix Riccati equation for discrete time systems.

$$M = Q + A' [M^{-1} + BR^{-1} B']^{-1} A \tag{2-26}$$

Remark 4.

In theorem II-1, only a necessary condition is derived and sufficiency is not proved. It is not clear that the solution of equation (2-13) combined with equations (2-14) and (2-15) is unique.

III. Optimal proportional-plus-integral feedback

In this section, an extension is made to a discrete version of output PI feedback control.

For summing action of the output, let's introduce an augmented r x 1 state vector z(k). Then the system can be described by

$$x(k+1) = A\,x(k) + B\,u(k) \tag{3-1}$$

$$y(k) = C\,x(k) \tag{3-2}$$

$$z(k+1) = z(k) + y(k) \tag{3-3}$$

To assure the open loop asymptotic stability of the system including the augmented states, let

$$z(k+1) = (1-\epsilon)\, z(k) + C\,x(k)$$

where ϵ is a small positive real number

In PI control the u(k) is constrained in the form of

$$u(k) = -F_p C\,x(k) - F_i\, z(k) \tag{3-4}$$

where F_p and F_i are m x r real constant matrices.

From equations (3-1) to (3-3), one can write the open loop state equation as

$$\hat{x}(k+1) = \hat{A}\,\hat{x}(k) + \hat{B}\,u(k) \tag{3-5}$$

where

$$\hat{A} = \begin{bmatrix} A & O \\ C & \hat{I} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ O \end{bmatrix}, \quad \hat{I} = I - \epsilon I$$

Combining equations (3-4) and (3-5), the closed loop state equation can be written in the form of

$$\hat{x}(k+1) = \begin{bmatrix} A - BF_pC & -BF_i \\ C & \hat{I} \end{bmatrix} \hat{x}(k) \tag{3-6}$$

Let

$$\hat{y}(k) = \hat{C}\,\hat{x}(k) \tag{3-7}$$

where

$$\hat{y}(k) = \begin{bmatrix} y(k) \\ z(k) \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & O \\ O & I \end{bmatrix}$$

Let $\hat{F} = [F_p : F_i]$

Now the performance criterion will be

$$\hat{J}_o = ½\, \mathrm{tr}\, [\sum_{k=o}^{\infty} \hat{A}_F'^{k} (\hat{Q} + \hat{C}'\hat{F}'\hat{R}\hat{F}\hat{C})\, \hat{A}_F^{k} \hat{x}(o)\hat{x}(o)'] \tag{3-8}$$

where

$$\hat{A}_F = \hat{A} - \hat{B}\hat{F}\hat{C} \tag{3-9}$$

$$\hat{x}(o) = \begin{bmatrix} x(o) \\ z(o) \end{bmatrix} \tag{3-10}$$

As before, we may average the performance criterion given by equation (3-8).

Let

$$\hat{X}_o = E[\hat{x}(o)\hat{x}(o)'] = \begin{bmatrix} X_o & O \\ O & Z_o \end{bmatrix}$$

$$\hat{Q} = \begin{bmatrix} Q_x & O \\ O & Q_z \end{bmatrix}$$

If we assume that the augmented states are initially at rest, Z_o will be a null matrix. Otherwise we can use Z_o as a design parameter together with X_o.

When choosing a weighting matrix $\hat{Q}$, it is recommended to choose a nonnull Q_Z, especially when C is a nonsingular matrix. If Q_Z is a null matrix and C is a nonsingular matrix, we can minimize $\hat{J}$ by using F_p only. In this case, F_i is a null matrix and integral (summing) action do not affect the performance of the system. However, in many cases, there may be step disturbances or uncertainties in system parameters. So if we want to keep the steady state error at near zero, integral action should be added to the controller. For this, Q_Z should not be a null matrix.

After averaging, equation (3-10) becomes,

$$\hat{J} = \tfrac{1}{2}\,\mathrm{tr}\,[\sum_{k=o}^{\infty} \hat{A}_{F'}{}^{k}\,(\hat{Q} + \hat{C}'\hat{F}'\hat{R}\hat{F}\hat{C})\,\hat{A}_F{}^{k}\,\hat{X}_o] \quad (3\text{-}11)$$

For system equations (3-5) and (3-7), and performance criterion given by equation (3-11), we can apply Theorm II-1 to get a necessary condition that must be satisfied by $\hat{F}$ for which $\hat{J}$ is minimum.

IV. Example

Consider a system repersented by

$$x(k+1) = \begin{bmatrix} .8 & .0 & .1 & .1 \\ .1 & .2 & .3 & .0 \\ .1 & .0 & .4 & .2 \\ .2 & .1 & .0 & .7 \end{bmatrix} x(k) + \begin{bmatrix} .0 & .0 \\ .1 & .2 \\ .0 & .3 \\ .4 & .0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0. & 1. & 0. & 0. \\ 0. & 0. & 0. & 1. \end{bmatrix} x(k)$$

$$x_o = \mathrm{diag}\,[\,1\ \ 1\ \ 1\ \ 1\,] = I$$

To solve this example, gradient method is used, i.e.

$$F_{k+1} = F_k - \alpha_k \nabla J(F_k)$$

where α_k is a positive real number.

1) Output proportional feedback

with Q = I (identity)

R = I

one can get,

$$F = \begin{bmatrix} .21385 & .72245 \\ .14112 & .29502 \end{bmatrix}$$

J = 4.473.

2) Output PI feedback

with $Q = \mathrm{diag}\,[\,1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\,]$

R = I

$Z_o = 0$

$$\hat{I} = I - \epsilon I = \begin{bmatrix} .9999 & .0 \\ .0 & .9999 \end{bmatrix}$$

one can get

$$\hat{F} = [F_p\ F_i] = \begin{bmatrix} .21100 & .71576 & .18950 & .49250 \\ .14058 & .29006 & .09200 & .25070 \end{bmatrix}$$

J = 4.460

In this case, $\hat{C}$ is taken as $\begin{matrix} C & 0 \\ O & 0.01\cdot I \end{matrix}$ only for fast convergence. By doing this, the overall characteristics do not change but the values of elements of F_i are multiplied by 100.

3) Optimal state feedback control

Let C = I, with same A, B, Q, and R matrices as in example 1). Then,

$$F = \begin{bmatrix} .39489 & .08079 & .09148 & .46657 \\ .17931 & .04466 & .21205 & .12382 \end{bmatrix}$$

J = 4.024 .

In the above example, the improvement of performance by introducting integral action to output proportional feedback control is rather small. But in real applications, integral action may compensate the modeling errors and steady state disturbances and will tend to make the steady state error go to near zero which is not possible for output proportional feedback control or optimal state feedback control.

V. Conclusion

Using the gradient of a standard quadratic performance criterion, a necessary condition for optimal output feedback gains is derived. Also, an extension is made to discrete version of output proportional plus integral feedback control. The PI control is simple and practical, because there are no needs to estimate or reconstruct the whole state, and because, due to integral action, the steady state error will go to zero, when there are errors in system parameters, provided that the closed loop system remains stable and time invariant.

References

1. William S. Levine and Michael Athans, On the determination of the optimal constant output feedback gains for linear multivariable systems, IEEE, vol. AC-15, pp. 44-48, Feb., 1970.
2. James. A. Heinen, A technique for solving the extended discrete Lyapunov Matrix Equation, IEEE Trans. on A.C., pp. 156-157, 1972.
3. S.S. Choi and H.R. Sirisena, Computation of optimal output feedback gains for linear multivariable systems, IEEE Trans. on A.C., pp. 257-258, 1974.
4. Hans P. Horisberger and Pierre R. Belanger, Solution of the optimal constant output feedback problem by conjugate gradients, IEEE Trans. on A.C., pp. 434-435, 1974.
5. Jerry M. Mendel, A concise derivation of optimal constant limited state feedback gains, IEEE Trans. on A.C., pp. 447-448, 1974.
6. C.S. Berger, An algorithm for designing suboptimal dynamic controllers, IEEE Trans. on A.C., pp. 596-597, 1974.
7. S.P. Bingulac, N.M. Cuk and M.S. Calovic, Calculation of optimum feedback gains for output-constrained regulators, IEEE Trans. on A.C., pp. 164-166, 1975.
8. DJ. B. Petkovski and M. Rakic, On the calculation of optimum feedback gains for output-constrained regulators, IEEE Trans. on A.C., pp. 760, 1978.
9. W.J. Hutcheson, A simple derivation of the gradient conditions for optimal constant output feedback gains, IEEE Trans. on A.C., pp. 937-938, 1978.
10. Gerald S. Mueller and Victor O. Adeniyi, Optimal output feedback by gradient methods with optimal stepsize adjustment, proc. of IEEE, 1979.
11. H. Seraji, Design of proportional-plus-integral controllers for multivariable systems, INT. J. Control, vol. 29, No. 1, pp. 49-63, 1979.
12. Peter Dorato and Alexander H. Levis, Optimal linear regulators: The discrete-time case, IEEE Trans. on AC., pp. 613-620, 1972.
13. A.P. Sage and C.C. White, III, Optimum systems control, 2nd Ed., Prentice-Hall, 1977.