論 文
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1部 : 정확한 閉型解

Optimal Control of a Nuclear Reactor with Distributed Parameters-Part I: Exact-Closed form Solution

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Abstract

The analytical treatment for a terminal cost problem of a distributed reactor with a small singular parameter is presented. The inverse of the neutron velocity is regarded as a singular parameter, and the model, adopted for simplicity, is a cylindrically symmetrical reactor. The Helmholtz mode expension is used for the application of the optimal theory for lumped parameter systems to the spatially distributed parameter system. The closed-form solution is explicitely obtained for machine calculation.

INTRODUCTION

Studies on the optimal control of nuclear reactor dynamics have been made in the past by many authors, but they have mostly treated the reactor system directly as a lumped parameter model. However, the recent development of large power reactors has called attention to studies of a class of optimal control problems invoving a distributed parameter reactor. A number of papers have been devoted to the subject of optimal control of systems with spatially distributed parameters via various approaches, that is, modal expansion, nodal expansion, function space method, dynamic programming, and so forth, a few of which are referenced.[1-8] However, for reactor systems containing a perturbing parameter whose presence changes the order of model i.e., the number of independent vaiables, only an approximate solution based the singular perturbation theory[9] and an exact closed-form solution[10] have bee developed. In these solutions, the flux distribution at the initial time are assumed to be the one at the steady state. Since the initial state in the general optimization problems is a dynamic state, these two solutions are of limited application.

In this paper, we consider a terminal cost problem of transfering any initial distribution of neutron flux to a desired one in a specilied time for the cylindrically symmetrical reactor. The reactor dynamics are described by a particular set of equations: the one-group diffusion equation with one delayed neutron group. The inverse of the neutron velocity is regarded as a singularly perturbing parameter. An exact solution for the control law can be written in the closed-form by using the Sylvester theorem[10,11] after the optimal control problem for the distributed parameter system is transformed to the lumped parameter system via the Helmholtz mode expansion.

THEORY

We consider here, for simplicity, a cylindrically symmetrical reactor of radius R and height H. The reactor dynamics are described by the following set of equations, the one-group diffusion equation with one delayed neutron group:

$$\frac{1}{v}\frac{\partial\phi}{\partial t}=\nabla\cdot D(\vec{r})\nabla\phi(\vec{r},t)-\sum_a(\vec{r})\phi(\vec{r},t)+$$
$$(1+\beta)\nu\Sigma^f(\vec{r})\phi(\vec{r},t)+\lambda C(\vec{r},t)-\Sigma_c^*(\vec{r},t)\phi(\vec{r},t), \qquad (1)$$

$$\frac{\partial C(\vec{r},t)}{\partial t}=\beta\nu\sum_f(\vec{r})\phi(\vec{r},t)-\lambda C(\vec{r},t), \qquad (2)$$

where

$\phi(\vec{r},t)$=neutron flux

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接受日字 : 1979年 11月 19日

$C(\vec{r},t)$=precursor density

$\Sigma_c^*(\vec{r},t)$=absorption cross section

wich is used for a control variable. The other notations have the usual meanings. This system is in bilinear form, since the product of the distributed control variable and neutron flux is involved. Hence, the linearization of this system is necessary for an expedient analytical treatment, However, the linearized equations obtained by expansion about any zeroth-order state do not have analytical eigenfunctions, since Eqs. (1) and (2) include spatially varying parameters. To avoid this situation, we separate the system parameters, i.e., $D(\vec{r})$, $\Sigma_a(\vec{r})$ and $\Sigma_f(\vec{r})$ into two spatially independent and spatially variant first-order small parts. The spatially independent zeroth-order parts are chosen such that $\phi_o(\vec{r})$ which satisfies

$$0=D^o\nabla^2\phi(\vec{r})-\Sigma_a^o\phi(\vec{r})+(1-\beta)\nu\Sigma_f^o\phi_o(\vec{r})+\lambda C_o(\vec{r}). \quad (3)$$

$$0=\beta\nu\Sigma_f^o\phi_o(\vec{r})-\lambda C_o(\vec{r}) \quad (4)$$

is equal to the fundamental component of the initial flux distribution, $\phi(\vec{r},0)$ in the Helmholtz mode expansion. Here, the superscript o indicates the zeroth-order. The spatially variant parts are taken into the control as the initial distribution of control.

A linearized first-order equations can be obtained by considering small deviations from $\phi_o(r)$ and $C_0(\vec{r})$ involving only spatially independent parameters as follows:

$$\overleftrightarrow{E}\frac{\partial\vec{\phi}(\vec{r},t)}{\partial t}=\overleftrightarrow{S}\vec{\phi}(\vec{r},t)+\vec{B}u(\vec{r},t)+\vec{D} \quad (5)$$

where

$$\vec{\phi}(\vec{r},t)=\begin{pmatrix}\delta\phi(\vec{r},t)\\ \delta C(\vec{r},t)\end{pmatrix},$$

$$\overleftrightarrow{E}=\begin{pmatrix}\epsilon & 0\\ 0 & 1\end{pmatrix},$$

$$\overleftrightarrow{B}=\begin{pmatrix}1\\ 0\end{pmatrix},$$

$$\overleftrightarrow{S}=\begin{pmatrix}D^o\nabla^2+(1-\beta)\Sigma_f^o-\Sigma_a^o, & \lambda\\ \beta\nu\Sigma_f^o & -\lambda\end{pmatrix},$$

$$u(\vec{r},t)=-\Sigma_c^*(\vec{r},t)+(1-\beta)\nu\delta\Sigma_f(\vec{r})-\delta\Sigma_a(\vec{r})+\frac{1}{\phi_o(\vec{r})}\nabla\cdot\delta D(\vec{r})\nabla\phi_o(\vec{r}),$$

$$\vec{D}=\begin{pmatrix}0\\ w(\vec{r})\end{pmatrix},$$

$$w(\vec{r})=\beta\nu\delta\Sigma_f(\vec{r})\phi_o(\vec{r}),\ \epsilon=\frac{1}{v}. \quad (6)$$

In the problem to be studied, the necessary optimal control will be obtained to change the neutron flux and precursor density distribution from a given initial distribution to the desired distribution in a specified time This time, T, is given in the terminal cost problem to minimize the following performance index:

$$J=\int_V[\vec{\phi}(\vec{r},T)-\vec{\phi}_d(\vec{r})]\overleftrightarrow{Q}[\vec{\phi}(\vec{r},T)-\vec{\phi}_d(\vec{r})]d^3r+\int_V\int_0^T r\,[u(\vec{r},t)-u_o(\vec{r})]^2d^3rdt \quad (7)$$

where

$$Q=\begin{pmatrix}q_1 & 0\\ 0 & q_2\end{pmatrix},\ q_1,\ q_2,\ r>0.$$

Here, $\phi_d(\vec{r})$ is the desired flux distribution and $u_o(\vec{r})$ is the initial value of the control. Note that the regulator problem of the linear system with terminal cost is mathematically equivalent to the problem stated here in that the deviations from the equilibrium in the regulator problem correspond to those from the desired state in the above problem. This is the case when the desired state is not far from the initial state.

MODAL EXPANSION

The Helmholtz mode in the cylindrically symmetrical reactor is defined by

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial z^2}+\mu\varphi=0, \quad (8)$$

with

$$\varphi(R,z)=0,\ \text{for}\ 0\le z\le H,$$
$$\varphi(r,0)=0,\ \text{for}\ 0\le r\le R,$$
$$\varphi(r,H)=0,\ \text{for}\ 0\le r\le R, \quad (9)$$

where R and H are the extrapolated radius and height, respectively. The eigenvalue $\mu_{n,l}$ and eigenfunction $\varphi_{n,l}$ are

$$\mu_{n,l}=\left(\frac{x_n}{R}\right)^2+\left(\frac{l\pi}{H}\right)^2 \quad (10)$$

and

$$\varphi_{n,l}=\frac{2}{\pi HR^2[J_1(x_n)]^2}\; J_0\left(\frac{x_n r}{R}\right)\sin\left(\frac{l\pi z}{H}\right) \tag{11}$$

where J_n is the Bessel function of order n and x_n is the root of $J_0(x)=0$.

The sequence $\{\varphi_{n,l}\}$ constitutes an orthonormal set, i.e.,

$$\int_V \varphi_{n,l}(\vec{r})\,\varphi_{n,l}(\vec{r})d^3r=\delta_{n,n'}\,\delta_{l,l'} \tag{12}$$

where δ is the Kroencker delta.

We can expand all space variables into an infinite series with respect to the eigenfunctions generated above as follows:

$$\vec{\varphi}(\vec{r},t)=\sum_{n,l}\begin{pmatrix}x_{n,l}(t)\\ y_{n,l}(t)\end{pmatrix}\varphi_{n,l}(\vec{r}), \tag{13}$$

$$\vec{\varphi}_d(\vec{r},t)=\sum_{n,l}\begin{pmatrix}x^d_{n,l}\\ y^d_{n,l}\end{pmatrix}\varphi_{n,l}(\vec{r}), \tag{14}$$

$$u(r,t)=\sum_{n,l}u_{n,l}(t)\phi_{n,l}(\vec{r}), \tag{15}$$

$$u_0(\vec{r})=\sum_{n,l}u^0_{n,l}\varphi_{n,l}(\vec{r}), \tag{16}$$

$$\vec{D}(\vec{r})=\sum_{n,l}\begin{pmatrix}0\\ w_{n,l}\end{pmatrix}\varphi_{n,l}(\vec{r}). \tag{17}$$

Substituting Eqs. (13) through (17) into Eq. (5) with use of the orthonormality, Eq. (12), yields

$$\epsilon\frac{dx_{n,l}(t)}{dt}=a_{n,l}x_{n,l}(t)+\lambda y_{n,l}(t)+u_{n,l} \tag{18}$$

and

$$\frac{dy_{n,l}(t)}{dt}=bx_{n,l}(t)-\lambda y_{n,l}(t)+w_{n,l}, \tag{19}$$

where

$$a_{n,l}=-D^0\mu_{n,l}+(1-\beta)\nu\Sigma^0_f-\Sigma^0_a$$

$$b=\beta\nu\Sigma^0_f \tag{20}$$

Similarly, we obtain the modal expansion of the performance index as follows:

$$J=\sum_{n,l}\left[\tilde{\vec{\phi}}_{n,l}(T)\overleftrightarrow{Q}\tilde{\vec{\phi}}_{n,l}(T)+r\int_0^T\tilde{u}^2_{n,l}(t)dt\right], \tag{21}$$

where

$$\left.\begin{aligned}\tilde{\vec{\phi}}_{n,l}(T)&=\vec{\phi}_{n,l}(T)-\vec{\phi}^d_n\\ \tilde{u}_{n,l}(t)&=u_{n,l}(t)-u^0_{n,l}\end{aligned}\right\} \tag{22}$$

Since coupling between each mode does not occur in the total system, as one readily finds from Eqs. (18), (19), and (21), the problem is reduced to find an admissible control $u^*(t)$ that causes the lumped parameter systems

$$\frac{d\vec{\phi}(t)}{dt}=\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\begin{pmatrix}\frac{a}{\epsilon} & \frac{\lambda}{\epsilon}\\ b & -\lambda\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}+\begin{pmatrix}\frac{1}{\epsilon}\\ 0\end{pmatrix}u(t)+\begin{pmatrix}0\\ w\end{pmatrix}, \tag{23}$$

to follow an admissible trajectory $\vec{\phi}(t)$ that minimizes the performance index

$$J=\frac{1}{2}\tilde{\vec{\phi}}(T)\overleftrightarrow{Q}\tilde{\vec{\phi}}(T)+\frac{1}{2}r\int_0^T\tilde{u}^2(t)dt \tag{24}$$

In Eqs. (23) and (24), the subscript of each variable representing the modal order is omitted and the same procedure is adopted in the following, where there is no possible chance of con fusion. The matrix equation, Eq. (23), does not appear to give an expedient approximate solution by applying the ordinary perturbation theory due to presence of the very small parameter ϵ. The so-called singular perturbation theory is applied for such a system to give the first-order approximation solution by Asatani et al.[9] This perturbation theory necessitates extremely cumbersome calculations for higher order calculations other than the first-order one. Furthermore, it is wholly impossible to treat the reactor systems described by the multigroup diffusion equations by this theory. A systematic nonperturbation technique that gives an exact closed-form solution is developed herein.

EXACT CLOSED-FORM SOLUTION

Applying the well-known results in optimal control theory of lumped parameter systems,[21] the following control law is derived in the form of feedback type:

$$\frac{d}{dt}\begin{pmatrix}x^*(t)\\ y^*(t)\\ p_1^*(t)\\ p_2^*(t)\end{pmatrix}=\begin{pmatrix}\frac{a}{\epsilon} & \frac{\lambda}{\epsilon} & -\frac{1}{r\epsilon^2} & 0\\ b & -\lambda & 0 & 0\\ 0 & 0 & -\frac{a}{\epsilon} & -b\\ 0 & 0 & -\frac{\lambda}{\epsilon} & \lambda\end{pmatrix}\begin{pmatrix}x^*(t)\\ y^*(t)\\ p_1^*(t)\\ p_2^*(t)\end{pmatrix}$$

$$+\begin{pmatrix} \frac{u_o}{\epsilon} \\ w \\ 0 \\ 0 \end{pmatrix}$$

where * denotes the optimal value. These equations should be solved under the initial and terminal conditions:

$$\left.\begin{aligned} &x_{1,1*}(0)=y_{1,1*}(0)=0 \\ &x_{n,l}{}^*(0)=\int_V[\phi(\vec{r},0)-\phi_o(\vec{r})]\phi_{n,l}(\vec{r})d^3r \\ &y_*{}^{n,l}(0)=\int_V[C(\vec{r},0)-C_o(\vec{r},0)]\phi_{n,l}(\vec{r})d^3r \end{aligned}\right\} \quad (26)$$

for $(n,l)=(1.1)$, and

$$\left.\begin{aligned} p_{1*}(T)=q_1[x^*(T)-x_d] \\ p_{2*}(T)=q_2[y^*(T)-y_d] \end{aligned}\right\} \quad (27)$$

The formal solution of Eq. (25) with the boundary conditions.

$$\begin{pmatrix} x^*(t) \\ y^*(t) \\ p_{1*}(t) \\ p_{2*}(t) \end{pmatrix}=\exp(\overleftrightarrow{K}t)\left\{\int_0^t\exp(-\overleftrightarrow{K}t')fdt'+\begin{pmatrix} x^*(0) \\ y^*(0) \\ p_{1*}(0) \\ p_{2*}(0) \end{pmatrix}\right\} \quad (28)$$

where

$$\overleftrightarrow{K}=\begin{pmatrix} \frac{a}{\epsilon} & \frac{\lambda}{\epsilon} & -\frac{1}{r\epsilon^2} & 0 \\ b & -\lambda & 0 & 0 \\ 0 & 0 & -\frac{a}{\epsilon} & -b \\ 0 & 0 & -\frac{\lambda}{\epsilon} & \lambda \end{pmatrix}, \quad (29)$$

$$\vec{f}=\begin{Vmatrix} u_o \\ w \\ 0 \\ 0 \end{Vmatrix} \quad (30)$$

By means of the Sylvester theorem, [10][11] the matrix functions $\exp(\overleftrightarrow{K}t)$ and $\exp(-\overleftrightarrow{K}t)$ can be written as follows:

$$\exp(\overleftrightarrow{K}t)=\overleftrightarrow{A}\exp(\lambda_1 t)+\overleftrightarrow{B}\exp(\lambda_2 t)+\overleftrightarrow{C}\exp(-\lambda_1 t)+\overleftrightarrow{D}\exp(-\lambda_2 t), \quad (31)$$

$$\exp(-\overleftrightarrow{K}t)=\overleftrightarrow{C}\exp(\lambda_1 t)+\overleftrightarrow{D}\exp\lambda_2 t)+\overleftrightarrow{A}\exp(-\lambda_1 t)+\overleftrightarrow{B}\exp(-\lambda_2 t), \quad (32)$$

where

$$\lambda_1=\frac{1}{2}(\alpha+\beta). \qquad \lambda_2=\frac{1}{2}(\alpha-\beta), \quad (33)$$

$$\overleftrightarrow{A}=\frac{(\overleftrightarrow{K}+\lambda_1\overleftrightarrow{I})(\overleftrightarrow{K}^2-\lambda_2^2\overleftrightarrow{I})}{\alpha\beta(\alpha+\beta)}$$

$$\overleftrightarrow{B}=\frac{(\overleftrightarrow{K}+\lambda_2\overleftrightarrow{I})(\overleftrightarrow{K}^2-\lambda_1^2\overleftrightarrow{I})}{\alpha\beta(\alpha-\beta)}$$

$$\overleftrightarrow{C}=\frac{(\overleftrightarrow{K}-\lambda_1\overleftrightarrow{I})(\overleftrightarrow{K}^2-\lambda_2^2\overleftrightarrow{I})}{\alpha\beta(\alpha+\beta)}$$

$$\overleftrightarrow{D}=\frac{(\overleftrightarrow{K}-\lambda_2\overleftrightarrow{I})(\overleftrightarrow{K}^2-\lambda_1^2\overleftrightarrow{I})}{\alpha\beta(\alpha-\beta)} \quad (34)$$

$$\left.\begin{aligned} &\alpha=a+b\epsilon \\ &\beta=\frac{1}{\epsilon}[a^2+(2ab-4\lambda^2)\epsilon+b^2\epsilon^2] \end{aligned}\right\} \quad (35)$$

and where I is the unit matrix.

Substitution of Eqs. (31) and (32) into Eq. (28) yields

$$Z_i(t)=\sum_{j=1}^4 U_{ij}(t)V_j(t)+U_{i1}(t)x^*(0)+U_{i2}(t)y^*(0)+U_{i3}(t)p_{1*}(0)+U_{i4}(t)p_{2*}(0) \quad (36)$$

where

$$Z_1=x^*,\ Z_2=y^*,\ Z_3=p_{1*},\ Z_4=p_{2*}, \quad (37)$$

$$\vec{U}=\exp(\overleftrightarrow{K}t), \quad (38)$$

$$V_j(t)=\frac{\exp(\lambda_1 t)-1}{\lambda_1}\left(\frac{u_o}{\epsilon}C_{j1}+wC_{j2}\right)+\frac{\exp(\lambda_2 t)-1}{\lambda_2}\left(\frac{u_o}{\epsilon}D_{j1}+wD_{j2}\right)-\frac{\exp(-\lambda_1 t)-1}{\lambda_1}\left(\frac{u_o}{\epsilon}A_{j1}+wA_{j2}\right)-\frac{\exp(-\lambda_2 t)-1}{\lambda_2}\left(\frac{u_o}{\epsilon}B_{j1}+wB_{j2}\right)$$

$$j=1,2,3,4. \quad (39)$$

In Eq. (36), $p_{1*}(0)$ and $p_{2*}(0)$ are not yet determined. These values are determined by Eq. (27), i.e.,

$$p_{1*}(0)=\frac{1}{G}\{G_1[U_{44}(T)-q_2U_{24}(T)]-G_2[U_{34}(T)-q_1U_{14}(T)]\} \quad (40)$$

$$p_{2*}(0)=\frac{1}{G}\{G_2[U_{33}(T)-q_1U_{13}(T)]-G_1[U_{43}(T)-q_2U_{23}(T)]\} \quad (41)$$

where

$$G=[U_{33}(T)-q_1U_{13}(T)][U_{44}(T)-q_2U_{24}(T)]-[U_{34}(T)-q_1U_{14}(T)][U_{43}(T)-q_2U_{23}(T)] \quad (42)$$

$$G_1=\sum_{j=1}^4[q_1U_{1j}(T)-U_{3j}(T)]V_j(T)+[q_1U_{11}(T)-U_{31}(T)]x^*(0)+[q_1U_{12}(T)-U_{32}(T)]y^*(0)-q_1x_d \quad (43)$$

$$G_2=\sum_{j=1}^4[q_2U_{2j}(T)-U_{4j}(T)]V_j(T)+[q_2U_{21}(T)-U_{41}(T)]x^*(0)+[q_2U_{22}(T)-U_{42}(T)y]^*(0)-q_2y_d. \quad (44)$$

The practical computation prohibits of the use the infinite terms of the modal expansion, and hence the truncated series should be adopted at a finite N'th mode.

REMARKS

The exact closed-form solution of an optimal problem, with terminal cost, arising in a distributed parametet nuclear reactor, has been derived. This method can be applied to the reactor sysyems described by the multigroup diffusion equations.

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