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A General Mixture-Truncated Method for Generating Bivariate Random Variables From Univariate Generators

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I. INTRODUCTION

In may simulation applications, it is required to generate bivariate random variables which have identical marginal distribution.

For example, in reliability problems, the assumption that two components have independent exponential failure times is very often unrealistic, and much effort has gone into deriving bivariate exponeitial random variables to handle these situations (Gaver (1972), Olkin & Marshall (1967), Downton (1970)). Now except in very specific physical situations it may be difficult to specify the complete bivariate distribution of life time of each component. However, it may be realistic to specify the marginal distributions and some measure of dependence (usually the correlation coefficient) between the life time of each component. In this kind of situation, we can use bivariate random vectors having given marginal distribution and dependence to solve the problem in simulation. To generate these vectors, there exists some previous works but most of these previous work is specific to specified marginal distributions and

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uses inverse transformation methods as a basic concept.

An example is the recent work by Johnson and Tenenbein (1979), to generate a bivariate random vector (X,Y) which has marginal distribution $F_1(x)$, $F_2(y)$ and correlation ρ, by a weighted linear combination method.

Define

$$X = F_1^{-1}(H_1(U))$$

$$Y = F_2^{-1}(H_2(V))$$

or

$$Y = F_2^{-1}(1 - H_2(V))$$

where H_1 and H_2 are the cumulative distribution functions (c.d.f.) of U and V respectively and

$$U = U'$$

$$V = cU' + (1 - c)V'$$

where U', V' are i.i.d. random variables with probability density function g( ). In this procedure, F_1^{-1}, F_2^{-1}, g( ) and C are specific to the marginal distribution and correlation desired. The functions F_1^{-1} and F_2^{-1} are difficult to compute in most cases and the weighting factor c is also difficult to calculate. Moreover most of the work in univariate random number generation has been aimed at avoiding having to calculate inverse cumulative distribution functions such as $F_1^{-1}(.)$ and $F_2^{-1}(.)$. These are the reasons why many proposed methods are specific to a specified marginal distribution. Again special properties of certain random variables such as infinite divisibility have been exploited to give easily generated bivariate random variables, often though with limited ranges of dependency. One very clever scheme by Gaver (1972) to generate bivariate exponential random variables uses the fact that the sum of a geometrically distributed number of exponential random variables (Y) is exponentially distributed and that the minimum of this geometrically distributed number of independent logistic random variables (Z) is exponential. Clearly when Y is large, Z is small. This scheme is of course very specific to exponential marginal distributions and, via an exponential transformation, to uniform random variables. To avoid these kinds of limitations and to make the generation of bivariate random variables simpler and more automatic in simulations, we develop here the general mixture truncation method which requires only that a method be available for generating random variables with the desired marginal distribution. Also we showed the procedure for generating bivariate exponential random variables as an example. Finally, I would like to note that this report is the part of my thesis for Master's Degree in Operations Research at the U.S. Naval Postgraduate School.

II. GENERAL MIXTURE-TRUNCATION METHOD

Denote by (Y,Z) the bivariate random pair, where each has identical marginal continuous distribution F(x), and denote a general random variable from this distribution by X. The argument is not specific to continuous random variables; this aspect comes in only in the computation of the correlations and can be developed in a parallel fashion for discrete marginal distributions. Let

$$P = \begin{bmatrix} \alpha_1 & 1-\alpha_1 \\ 1-\alpha_2 & \alpha_2 \end{bmatrix}$$

with stationary vectors

$$\bar{\pi} = \bar{\pi} P = \left(\frac{1-\alpha_2}{1-\alpha_1+1-\alpha_2}, \frac{1-\alpha_1}{1-\alpha_1+1-\alpha_2} \right) = (\pi_1, \pi_2),$$

and let X_1 be an X truncated to the left of a fixed point x_0, X_2 be an X truncated to the right of x_0, so that

$$F_{X_1}(x) = P[X_1 \le x] = \begin{cases} \dfrac{F(x)}{F(x_0)} & \text{if} \quad x \le x_0 \\ 1 & \text{if} \quad x > x_0 \end{cases}$$

$$F_{X_2}(x) = P[X_2 \le x] = \begin{cases} 0 & \text{if} \quad x \le x_0 \\ \dfrac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if} \quad x > x_0 \end{cases}$$

In addition we set $\pi_1 = F(x_0)$, $\pi_2 = 1 - \pi_1$ and choose Y and Z as follows.

i) Choose Y from X_1 with probability π_1 and then choose Z from X_1 with probability α_1, or from X_2 with probability $1 - \alpha_1$.

ii) Choose Y from X_2 with probability π_2, where $\pi_1 + \pi_2 = 1$, and then choose Z from X_1 with probability $1-\alpha_2$, or from X_2 with probability α_2.

If we choose (Y,Z) as in the above procedure, then we can make the following two theorems.

A. MARGINAL DISTRIBUTIONS

Theorem 1.

The marginal distribution of (Y,Z) becomes F(x) for both Y and Z.

Proof

1. Marginal distribution of Y

By definition Y is the mixture of X_1 and X_2 with probability π_1, π_2 respectively. That is

$$F_Y(x) = \pi_1 F_{X_1}(x) + \pi_2 F_{X_2}(x)$$

$$F_Y(x) = \begin{cases} \pi_1 \dfrac{F(x)}{F(x_0)} + \pi_2 \cdot 0 & \text{if} \quad x \leq x_0 \\ \pi_1 \cdot 1 + \pi_2 \dfrac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if} \quad x > x_0 \end{cases}$$

But since we define $\pi_1 = F(x_0)$: $\pi_2 = 1 - \pi_1 = 1 - F(x_0)$, we have

$$F_Y(x) = \begin{cases} \pi_1 \dfrac{F(x_0)}{\pi_1} = F(x) & \text{if} \quad x \leq x_0 \\ \pi_1 + \pi_2 \dfrac{F(x) - \pi_1}{\pi_2} = F(x) & \text{if} \quad x > x_0 \end{cases}$$

$= F(x)$ in all cases.

So Y has the marginal distribution $F(x)$.

2. Marginal distribution of Z

If Y is from X_1, then

$$F_{Z_1}(x) = \alpha_1 F_{X_1}(x) + (1 - \alpha_1) F_{X_2}(x)$$

$$= \begin{cases} \alpha_1 \dfrac{F(x)}{F(x_0)} + (1 - \alpha_1) \cdot 0 & \text{if} \quad x \leq x_0 \\ \alpha_1 \cdot 1 + (1 - \alpha_1) \dfrac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if} \quad x > x_0 \end{cases}$$

if Y is from X_2, then

$$F_{Z_2}(x) = (1 - \alpha_2) F_{X_1}(x) + \alpha_2 F_{X_2}(x)$$

$$= \begin{cases} (1-\alpha_2)\ \dfrac{F(x)}{F(x_0)} + \alpha_2 \cdot 0 & \text{if} \quad x \leq x_0 \\ (1-\alpha_2) \cdot 1 + \alpha_2\ \dfrac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if} \quad x > x_0 \end{cases}$$

so,

$$F_Z(x) = \pi_1 F_{Z_1}(x) + \pi_2 F_{Z_2}(x)$$

$$= \begin{cases} \pi_1 \alpha_1 \dfrac{F(x)}{F(x_0)} + \pi_2 (1-\alpha_2) \dfrac{F(x)}{F(x_0)} & \text{if} \quad x \leq x_0 \\ \pi_1 (\alpha_1 + (1-\alpha_1) \dfrac{F(x) - F(x_0)}{1 - F(x_0)} & \text{if} \quad x > x_0 \end{cases}$$

$$+ \pi_2 ((1-\alpha_2) + \alpha_2 \dfrac{F(x) - F(x_0)}{1 - F(x_0)}$$

and we defined π_1 and π_2 as follows.

$$\pi_1 = \frac{1-\alpha_2}{1-\alpha_1+1-\alpha_2}$$

$$\pi_2 = \frac{1-\alpha_1}{1-\alpha_1+1-\alpha_2}$$

From this, we know

$$1-\alpha_2 = \frac{\pi_1}{\pi_2}(1-\alpha_1)$$

If we use this relationship, then $F_Z(x) = F(x)$ in both cases. So Z also has the marginal distribution F(x). The result is a consequence of the fact $\underline{\pi}$ is defined to be the stationary vector associated with $\underline{P}$.

B. THE PRODUCT-MOMENT CORRELATION

Theorem 2.

The correlation coefficient between Y and Z becomes

$$\rho = \beta M,$$

where

$$-1 \leq \beta = \alpha_1 - (1 - \alpha_2) \leq 1,$$

and

$$M = (\mu_1 - \mu_2)^2 \pi_1 \pi_2 / \sigma_X^2,$$

where

$$\mu_1 = \int_0^{x_0} x \, d\frac{F(x)}{F(x_0)}$$

$$\mu_2 = \int_{x_0}^{\infty} x \, d\frac{F(x) - F(x_0)}{1 - F(x_0)}$$

$$\sigma_1^2 = \int_0^{x_0} x^2 \, d\frac{F(x)}{F(x_0)} - \mu_1^2$$

$$\sigma_2^2 = \int_{x_0}^{\infty} x^2 \, d\frac{F(x) - F(x_0)}{1 - F(x_0)} - \mu_2^2$$

$$\sigma_X^2 = \int_0^{\infty} x^2 \, dF(x) - E[X]$$

$$= \sigma_1^2 \pi_1 + \sigma_2^2 \pi_2 + (\mu_1 - \mu_2)^2 \pi_1 \pi_2$$

Proof

$$\rho_{YZ} = \frac{\mathrm{cov}[Y, Z]}{\sigma_Y \sigma_Z}$$

$$= \frac{E[YZ] - E[Y] \, E[Z]}{\sigma_Y \sigma_Z}$$

Now

$$E[YZ] = E_S[E[YZ \mid Y, Z \varepsilon S]]$$
$$= E[YZ \mid Y \varepsilon X_1, Z \varepsilon X_1] P[Y \varepsilon X_1, Z \varepsilon X_1]$$
$$+ E[YZ \mid Y \varepsilon X_1, Z \varepsilon X_2] P[Y \varepsilon X_1, Z \varepsilon X_2]$$
$$+ E[YZ \mid Y \varepsilon X_2, Z \varepsilon X_1] P[Y \varepsilon X_2, Z \varepsilon X_1]$$
$$+ E[YZ \mid Y \varepsilon X_2, Z \varepsilon X_1] P[Y \varepsilon X_2, Z \varepsilon X_2]$$
$$= E[X_1] E[X_1] \pi_1 \alpha_1 + E[X_1] E[X_2] \pi_1 (1-\alpha_1)$$
$$+ E[X_2] E[X_1] \pi_2 (1-\alpha_2) + E[X_2] E[X_2] \pi_2 \alpha_2$$
$$= \mu_1^2 \alpha_1 \pi_1 + \mu_1 \mu_2 (1-\alpha_1) \pi_1$$
$$+ \mu_1 \mu_2 (1-\alpha_2) \pi_2 + \mu_2^2 \alpha_2 \pi_2$$

where

$$\mu_1 = E[X_1], \quad \mu_2 = E[X_2]$$

Further,

$$E[Y] = E_S[E[Y \mid Y \varepsilon S]]$$
$$= E[Y \mid Y \varepsilon X_1] P[Y \varepsilon X_1] + E[Y \mid Y \varepsilon X_2] P[Y \varepsilon X_2]$$
$$= E[X_1] \pi_1 + E[X_2] \pi_2$$
$$= \mu_1 \pi_1 + \mu_2 \pi_2 = E[Z]$$

by Theorem 1.

Also by Theorem 1.

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2$$
$$= \sigma_X^2 = \sigma_Z^2$$

If we put together these formulae into

$$\rho_{Y,Z} = \frac{E[YZ] - E[Y] E[Z]}{\sigma_Y \sigma_Z}$$

we get

$$\rho_{Y,Z} = \frac{(\mu_1 - \mu_2)^2 \pi_1 \pi_2 (\alpha_1 - (1-\alpha_2))}{\sigma_X^2}$$

and let

$$\beta = \alpha_1 - (1 - \alpha_2)$$
$$M = (\mu_1 - \mu_2)^2 \pi_1 \pi_2 / \sigma_X^2$$

then

$$\rho = \beta M.$$

C. GENERAL ALGORITHM

We give here three algorithms for implementing the bivariate mixture-truncation method, which we call the FXO method, the UXO method, and the TXO method. All of these methods are exactly the same except in how the algorithm chooses x_0, the truncation point, from the x_0 range (x_ℓ , x_u). The first procedure, called the FXO, chooses x_0 as a fixed point from the range $[X_\ell, X_u]$ and uses the same X_0 during the entire routine. The second procedure, called the UXO, chooses x_0 uniformly from (x_ℓ, x_u) and repeats this step in every routine of the algorithm. The third procedure, called the TXO, is the same as the UXO procedure except in that it uses a triangular distribution instead of uniform. It is necessary to fix these choices of x_0 becuase in general there is more than one x_0 which will give a bivariate pair (Y,Z) with the given marginal distribution and given correlation. The first procedure, FXO, is defective in terms of their discontinuity of distribution while the second and the third, UXO and TXO, are satisfactory in this respect. The choice of the midpoint of the interval (x_ℓ, x_u) for x_0 in FXO is based on simulation experience . Note that the algorithm described here is inefficient in that it generates the truncated variables X_1 and X_2 by comparing random variables X to X_0 until one which is respectively greater than or less than X_0 is found. More efficient methods can be found in special cases such as the exponential, but the present algorithm requires only a generation of univariate random variables X without regard to the method used to do this. Of course initialization is required and this is specific to each marginal distribution.

General Mixture-Truncation Method

1. (Initialization)

   i) For given marginal distribution F(x) and correlation coefficient ρ find x_o ranges (x_ℓ, x_u)

2. Define truncation point x_o

   * FXO method

     i) $X_o = \frac{1}{2}(x_\ell + x_u)$

   * UXO method

     i) Generate a uniform (0,1) random variable V_1

     ii) $x_o = x_\ell + (x_u - x_\ell) * V_1$

   * TXO method

     i) Generate two uniform (0,1) random variables V_1, V_2.

     ii) $X_o = x_\ell + x_1 + x_2$

     where

$$x_m = \tfrac{1}{2}(x_\ell + x_u)$$

$$x_1 = (x_m - x_\ell) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

3. Compute parameters value, π_1, π_2, α_1, α_2

4. Choose type for Y

i) Generate a uniform (0,1) random variable U

ii) If $U \le \pi_1$, go to 9

5. Y is an X_2

i) Generate a random variable X from F(x)

ii) If $X > x_o$, set $Y \leftarrow X$ and go to 6

iii) Otherwise return to 5. i)

6. Choose type for Z

i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

ii) If $U \le 1 - \alpha_2$, go to 8

7. Z is an X_2

i) Generate a random variable X from F(x)

ii) If $X > x_o$, set $Z \leftarrow X$ and go to 11

iii) Otherwise return to 7. i)

8. Z is on X_1

i) Generate a random variable X from F(x)

ii) If $X \le x_o$, set $Z \leftarrow X$ and go to 11

iii) Otherwise return to 8. i)

9. Y is an X_1

i) Generate a random variable X from F(x)

ii) If $X \le x_o$, set $Y \leftarrow X$ and go to 10

iii) Otherwise return to 9. i)

10. Choose type for Z

i) Set $U \leftarrow U/\pi_1$

ii) If $U \le {}_1$, go to 8

iii) Otherwise go to 7

11. Deliver (Y,Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO method until a sufficient number of random vectors are obtained.

D. BIVARIATE DISTRIBUTION FUNCTIONS

From Theorem 1, in Secon II-A, we know that if Y is from X_1, then

$$F_Z(z \mid y) = \alpha_1 F_{X_1}(z) + (1-\alpha_1) F_{X_2}(z)$$

$$= \begin{cases} \alpha_1 \dfrac{F(z)}{F(x_0)} & \text{if} \quad z \leq x_0 \\ \alpha_1 + (1-\alpha_1) \dfrac{F(z) - F(x_0)}{1 - F(x_0)} & \text{if} \quad z > x_0 \end{cases}$$

and if Y is from X_2, then

$$F_Z(z \mid y) = (1-\alpha_2) F_{X_1}(z) + \alpha_2 F_{X_2}(z)$$

$$= \begin{cases} (1-\alpha_2) \dfrac{F(z)}{F(x_0)} & \text{if} \quad z \leq x_0 \\ (1-\alpha_2) + \alpha_2 \dfrac{F(z) - F(x_0)}{1 - F(x_0)} & \text{if} \quad z > x_0 \end{cases}$$

By using these we can define the bivariate distribution function as follows.

$$F(y,z) = P[Y \leq y, Z \leq z]$$
$$= \int_{-\infty}^{Y} P[Z \leq z \mid Y = u] \, dp[Y \leq u]$$

where

$$P[Y \leq u] = F(u)$$

$$P[Z \leq z \mid Y = u] = \begin{cases} \alpha_1 \dfrac{F(z)}{F(x_0)} & \text{if } u \leq x_0, z \leq x_0 \\ \alpha_1 + (1-\alpha_1) \dfrac{F(z)-F(x_0)}{1-F(x_0)} & \text{if } u \leq x_0, z > x_0 \\ (1-\alpha_2) \dfrac{F(z)}{F(x_0)} & \text{if } u > x_0, z \leq x_0 \\ (1-\alpha_2) + \alpha_2 \dfrac{F(z)-F(x_0)}{1-F(x_0)} & \text{if } u > x_0, z > x_0 \end{cases}$$

so, if we put these together, integrating with respect to $dp[Y \leq u]$, we get the final result :

$$P[Y \leq y, Z \leq z] = \begin{cases} \dfrac{\alpha_1}{\pi_1} F(z) F(y) & \text{if } y \leq x_0, z \leq x_0 \quad \text{(II-D-1)} \\ \left\{\alpha_1 + \dfrac{(1-\alpha_1)}{\pi_2} [F(z) - F(x_0)]\right\} F(y) & \text{if } y \leq x_0, z > x_0 \quad \text{(II-D-2)} \\ \left\{\alpha_1 + \dfrac{(1-\alpha_1)}{\pi_2} [F(y) - F(x_0)]\right\} F(z) & \text{if } y > x_0, z \leq x_0 \quad \text{(II-D-3)} \\ \alpha_1 \pi_1 + (1-\alpha_1) [F(z) - F(x_0)] \dfrac{\pi_1}{\pi_2} & \quad \text{(II-D-4)} \end{cases}$$

$$+\{(1-\alpha_2)+\frac{\alpha_2}{\pi_2}[F(z)-F(x_0)]\}[F(y)-F(x_0)] \quad \text{if } y>x_0, z>x_0$$

For example, the expression (II -D-4) is obtained as

$$F(y,z) = \int_{-\infty}^{y} P[Z\leq z \mid Y=u]\, dF(u),\ y > x_0,\ z > x_0$$

$$= \int_{-\infty}^{x_0} P[Z\leq z \mid Y=u\leq x_0]\, dF(u)$$

$$+ \int_{x_0}^{y} P[Z\leq z \mid Y=u> x_0]\, dF(u)$$

$$= \{\alpha_1 + \frac{(1-\alpha_1)}{\pi_2}[F(z)-F(x_0)]\} F(x_0)$$

$$+ (1-\alpha_2) + \frac{\alpha_2}{\pi_2}[F(z)-F(x_0)]\}[F(y)-F(x_0)]$$

It is easily seen from (II -D-2) that when $z\to\infty$, $P[Y\leq y, Z\leq\infty]=F(y)$: from (II -D-3) that as $y\to\infty$, $P[Y\leq\infty, Z\leq z]=F(z)$ and from (II - D - 4) that as $y\to\infty$, $P[Y\leq\infty, Z\leq z]=F(z)$ and that $z\to\infty$, $P[Y\leq y, Z\leq\infty]=F(y)$.
In particular from (II - D - 4) we have that, as $y\to\infty$,

$$F(y,z) \to \alpha_1 F(x_0) + (1-\alpha_2)[F(z)-F(x_0)] + (1-\alpha_2)\pi_2$$

$$+\alpha_2[F(z)-F(x_0)]$$

$$= \alpha_1 F(x_0) + (1-\alpha_2)\pi_2 + [F(z)-F(x_0)]$$

$$= F(z) + (1-\alpha_2)\pi_2 - (1-\alpha_1)\pi_1 = F(z)$$

where at the last step we used the facts that $F(x_0)=\pi_1$ and $\pi_1(1-\alpha_1)=\pi_2(1-\alpha_2)$. If $F(x)$ is absolutely continuous with probability density function $f(x)$ then the joint p.d.f. for the bivariate pair (Y,Z) is

$$f(y,z)=\begin{cases} \frac{\alpha_1(1-\alpha_1)}{\pi_2(1-\alpha_2)} f(y)f(z) & \text{if } y\leq x_0,\ z\leq x_0 \quad \text{(II -D-5)} \\ \frac{1-\alpha_1}{\pi_2} f(y)f(z) & \text{if } y\leq x_0,\ z> x_0 \quad \text{(II-D-6)} \\ \frac{1-\alpha_1}{\pi_2} f(y)f(z) & \text{if } y> x_0,\ z\leq x_0 \quad \text{(II -D-7)} \\ \frac{\alpha_2}{\pi_2} f(y)f(z) & \text{if } y> x_0,\ z> x_0 \quad \text{(II -D-8)} \end{cases}$$

Note that there is a discontinuity in the density function as one crosses the boundaries of the four quadrants defined by the lines $y = x_0$, $z = x_0$. The density is the same in the first and third quadrants. The multipliers of $f(y)\ f(z)/\ \pi_2$ are the same in all four quadrants iff there is independence. This occurs when $(1 - \alpha_1) = \alpha_2$ so that $\pi_1 = \alpha_2$ and $f(y,z) = f(y)f(z)$ for the whole range of y and z.

III. BIVARIATE EXPONENTIAL GENERATOR

Bivariate exponential random variable is one of the most interest random variable in simulations. The cumulative distribution function and probability density function for the exponential are, respectively,

$$F(X) = 1 - e^{-\lambda x} \qquad x \geq 0$$

and

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0$$

The expected value of the exponential distribution is

$$E[X] = 1/\lambda$$

and the variance is

$$VAR[X] = 1/\lambda^2.$$

The problem of generating exponential deviates reduces to one of generating "unit" exponentials, i.e., those with $\lambda = 1.0$, and then multiplying the result by whichever $1/\lambda$ is necessary to give the desired distribution. That is, if the random variable E has the exponential ($\lambda = 1$) distribution, then X defined as $E * \frac{1}{n}$ also has the exponential distribution with $\lambda = n$. Thus, in this section, we will consider only unit exponentials as a marginal distribution for bivariate pairs.

A. DETERMINATION OF PARAMETERS IN THE EXPONENTIAL MIXTURE-TRUNCATION METHOD

Because X_1 is an exponential ($\lambda = 1$) truncated to the left of x_o and X_2 is also exponential ($\lambda = 1$) truncated to the right of x_o, we have

$$E[X_1] = \mu_1 = \int_0^{x_o} x \, d \frac{F(x)}{F(x_o)}$$

$$= 1 - \frac{\pi_2}{\pi_1} x_o ;$$

$$VAR[X_1] = \sigma_1^2 = \int_0^{x_o} x^2 \, d \frac{F(x)}{F(x_o)} - \mu_1^2$$

$$= 1 - \frac{\pi^2}{\pi_1^2} \cdot X_0^2 ;$$

$$E[X_2] = \mu_2 = \int_{x_0}^{\infty} x \, d \frac{F(x) - F(x_0)}{1 - F(x_0)}$$

$$= 1 + x_0 ;$$

$$VAR[X_2] = \int_{x_0}^{\infty} x^2 \, d \frac{F(x) - F(x_0)}{1 - F(x_0)} - \mu_2^2$$

$$= 1 ;$$

and from definition,

$$\pi_1 = \frac{1 - \alpha_2}{1 - \alpha_1 + 1 - \alpha_2}$$

$$= F(x_0)$$

$$= 1 - e^{-x_0} , \qquad \text{(III-A-1 a)}$$

$$\pi_2 = \frac{1 - \alpha_1}{1 - \alpha_1 + 1 - \alpha_2}$$

$$= 1 - F(x_0)$$

$$= e^{-x_0} . \qquad \text{(III-A-1 b)}$$

If we use these formulas in Theorem 2 in Section II, we get

$$M = \frac{\pi_2}{\pi_1} x_0^2 \qquad \text{(III-A-2)}$$

$$\beta = \frac{1}{\pi_2} (\alpha_1 - \pi_1) \qquad \text{(III-A-3)}$$

and

$$\rho = \frac{\alpha_1 - \pi_1}{\pi_1} x_o^2 \qquad \text{(III-A-4)}$$

For given correlation coefficient ρ, we can compute α_1 as a function of x_o from the formula (III-A-4) as

$$\alpha_1 = \frac{\rho \pi_1}{x_o^2} + \pi_1$$

$$= (1 + \frac{\rho}{x_o^2})(1 - e^{-x_o})$$

and we know that $\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$ from the formulas (III-A-1a) and (III-A-1b). From this,

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$$

$$= e^{-x_o} + e^{x_o}(1 - e^{-x_o})^2 \frac{\rho}{x_o^2}$$

These α_1 and α_2 are probabilities, so they have to be greater than or equal to zero and less than or equal to one;

$$0 \leq (1 - e^{-x_o})(1 + \frac{\rho}{x_o^2}) \leq 1 \qquad \text{(III-A-5a)}$$

$$0 \leq e^{-x_o} + e^{x_o}(1 - e^{-x_o})^2 \frac{\rho}{x_o^2} \leq 1 \qquad \text{(III-A-5b)}$$

From these two inequality equations, (III-A-5a) and (III-A5b), we can find the x_o ranges for given correlation coefficient ρ. To solve these equations, we can divide into two cases, one for positive correlation case and the other for negative correlation case. If correlation coefficient ρ is positive, then both equations are always positive. Thus we only need to find the x_o ranges which makes (III-A-5a) and (II-A-5b) are less than 1. Form the equation (III-A-5a), for the π_1 case,

$$\alpha_1 = (1 - e^{-x_o})(1 + \frac{\rho}{x_o^2}) \leq 1$$

becomes

$$\rho(e^{x_o} - 1) \leq x_o^2$$

and let

$$y_1 = x_o^2, \quad Y_2 = \rho(e^{x_o} - 1).$$

Because of the first derivatives of y_1 and y_2 are always positive, we know that these two functions are monotone increasing functions. Thus we can find x_o ranges which satisfy $y_1 \geq y_2$ by the Newton Raphson method. When using the Newton Raphson method, let $y = y_1 - y_2 = 0$ and find an approximate solution, by approximating exponential series, which we can use as a starting point. That is,

$$y = y_1 - y_2 = x_o^2 - \rho(1 + x_o + \frac{x_o^2}{2!} + \frac{x_o^3}{3!} - 1)$$

$$= \frac{\rho}{6} x_o^2 - (1 - \frac{\rho}{2}) x_o + \rho = 0$$

Then

$$x_0 = \frac{(1-\frac{\rho}{2}) \pm (1-\rho-\frac{5}{12}\rho^2)^{1/2}}{1/3\,\rho}$$

Starting with this approximate value in the Newton Raphson method, we can find x_o range, say (x_{ℓ_1}, x_{u_1}), which satisfies $0 \leq \alpha_1 \leq 1$. And for the α_2 case,

$$\alpha_2 = e^{-x_0} + e^{-x_0}(1-e^{-x_0})^2 \frac{\rho}{x_0^2} \leq 1$$

becomes

$$\rho(e^{x_0}-1) \leq x_0^2$$

This result is exactly the same as the α_1 case, that is, at the same range α_1 and α_2 satisfy constraints $\alpha_1 \leq 1$ and $\alpha_2 \leq 1$. Thus we can use x_{ℓ_1} and x_{u_1} as the lower bound of x_0, x_ℓ, and the upper bound of x_o, x_u. If correlation coefficient ρ is negative, then the equations (III-A-5a) and (III-A-5b) are always less than 1. Therefore we need to consider only one constraint which makes $\alpha_1 \geq 0$, $\alpha_2 \geq 0$. From the α_1 equation (III-A-5A), solve the inequality equation

$$0 \leq \alpha_1 = (1-e^{-x_0})(1+\frac{\rho}{x_0^2})$$

since $1 - e^{-x_o}$ is always positive, we see that to satisfy the inequality $1 + \frac{\rho}{x_o^2}$ should be positive, i.e.,

$$\frac{\rho}{x_o 2} \geq -1$$

equivalently, we have

$$x_o \geq \sqrt{-\rho}$$

In the α_2 case, from equation (III-A-5b),

$$0 \leq \alpha_2 = e^{-x_o} + e^{x_o} (1 - e^{-x_o})^2 \frac{\rho}{x_o 2}$$

or, equivalently, we have

$$x_o 2 \geq -\rho (e^{x_o} - 1)^2$$

As in the positive correlation case, we can find a starting point by approximation to solve this equation by Newton Raphson. The result comes out as

$$x_s = (\sqrt{-\rho} + \rho) / (-\frac{\rho}{2})$$

with this starting point we find another bound of x_o which satisfies $0 \leq \alpha_2$. This becomes the upper bound of x_o, x_u, and from the α_1 case, we have a constraint $x_o \geq \sqrt{-\rho}$ which becomes the lower bound of x_o, i.e., $x_\ell = \sqrt{-\rho}$. The lowest and highest correlations available for bivariate exponential pairs in mixture-truncation method are approximately -0.480 and 0.647 respectively. By comparison note that the most negative correlation available for bivariate exponential pairs with identical fixed marginals is $1 - \frac{\pi^2}{6}$ (More (1967)). Gaver's (1972) negatively correlated pair has correlations in the range (-0.5, 0). The table (III-1) shows the lower and upper bound of x_o in the mixture-truncation method with identical marginal exponential and given correlation.

Table III-I : The lower bound and upper bound of X in the mixture-truncation method with identical exponential marginal distributions

x_o range for each correlation

ρ	x_L	x_u	ρ	x_L	x_u
0.1	0.106	5.832	-0.1	0.317	1.984
0.2	0.225	4.723	-0.2	0.448	1.439
0.3	0.362	3.990	-0.3	0.548	1.103
0.4	0.527	3.395	-0.4	0.633	0.855
0.5	0.741	2.842	-0.45	0.671	0.751
0.6	1.082	2.223			

B. GENERATING PROCEDURE

We developed here all of three procedures, the FXO method, the UXO method, the TXO method for generating bivariate random vectors whose marginal distributions are unit exponential and correlation coefficient is ρ

As we mentioned in section II, all of these methods are exactly the same except in how X_o is choose from the x_o range (x_ℓ , x_u). And we also showed an efficient procedure for generating X_1 and X_2 This Efficient procedure can generate X_1 and X_2 directly instead of comparing random variables X to x_o until one which is respectively greater than or less than x_o is found.

Exponential Mixture-Truncation Method

1. (Initialization)

   i) For given $-0.48 < \rho < 0.64$, find x_ℓ and x_u

2. Define truncation point x_o

   * FXO method

     i) $x_o = \frac{1}{2}(x_\ell + x_u)$

   * UXO method

     i) Generate a uniform (0,1) random variable U_1

     ii) $x_o = x_\ell + (x_u - x_\ell) * U_1$

   * TXO method

     i) Generate two uniform (0,1) random variables V_1, V_2

     ii) $x_o = x_\ell + x_1 + x_2$

     where

$$x_m = (x_\ell + x_u)/2$$

$$x_1 = (x_m - x_\ell) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

3. Compute parameter values

$$\pi_1 = F(x_o) = 1 - e^{-x_o}$$

$$\pi_2 = 1 - \pi_1$$

$$\alpha_1 = \pi_1 \left(1 + \frac{\rho}{x_o^2}\right)$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1 - \alpha_1)$$

4. Choose type for Y

   i) Generate a uniform (0,1) random variable U

   ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

   i) Generate an exponential random variable E_1

   ii) If $E_1 > x_o$, set $Y \leftarrow E_1$ and go to 6

   iii) Otherwise, return to 5. i)

6. Choose type for z

   i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

   ii) If $U \leq 1 - \alpha_2$, go to 8

7. z is an X_2

   i) Generate an exponential random variable E_2

   ii) If $E_2 > x_o$, set $Z \leftarrow E_2$ and go to 11

   iii) Otherwise return to 7. i)

8. Z is an X_1

   i) Generate an exponential random variable E_2

   ii) If $E_2 \leq x_o$, set $Z \leftarrow E_2$ and go to 11

   iii) Otherwise return to 8. i)

9. Y is an X_1

i) Generate an exponential random variable E_1

ii) If $E_1 \leq X_o$, set $Y \leftarrow E_1$ and go to 10

iii) Otherwise return to 9. i)

10. Choose type for Z

i) Set $U \leftarrow U/\pi_1$

ii) If $U \leq \alpha_1$, go to 8

iii) Otherwise go to 7

11. Deliver (Y,Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO methods until a sufficient number of random vectors are obtained.

For the exponential case it is possible to give a more efficient algorithm in which X_1 and X_2 are generated exactly. The algorithm is as follows.

Efficient Exponential Mixture-Truncation Method

1. (Initialization)

i) For given $-0.48 < \rho < 0.64$, find x_ℓ and x_u

2. Define the truncation point x_o

* FXO method

$x_o = \frac{1}{2}(x_\ell + x_u)$

* UXO method

i) Generate a uniform (0,1) random variable U_1

ii) $x_o = x_\ell + (x_u - x_\ell) * U_1$

* TXO method

i) Generate two uniform (0,1) random variables V_1, V_2

ii) $x_o = x \quad + x_1 + x_2$

where

$$x_m = (x_\ell + x_u)/2$$

$$x_1 = (x_m - x_\ell) * V_1$$

$$x_2 = (x_u - x_m) * V_2$$

3. Compute parameter values

$$\pi_1 = F(x_o) = 1 - e^{-x_o}$$

$$\pi_2 = 1 - \pi_1$$

$$\alpha_1 = \pi_1 (1 + \frac{\rho}{x_o^2})$$

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2} (1 - \alpha_1)$$

4. Choose type for Y

i) Generate a uniform (0,1) random variable U

ii) If $U \leq \pi_1$, go to 9

5. Y is an X_2

i) Generate an exponential random variable E_1

ii) Set $Y \leftarrow x_o + E_1$

6. Choose type for Z

   i) Set $U \leftarrow ((U - \pi_1)/(1 - \pi_1))$

   ii) If $U \leq 1 - \alpha_2$, go to 8

7. Z is an X_2

   i) Generate an exponential random variable E_2

   ii) Set $Z \leftarrow x_0 + E_2$ and go to 11

8. Z is an X_1

   i) Generate a uniform (0, 1) random variable W_2

   ii) Set $Z \leftarrow - \ln (1.0 - W_2 * \pi_1)$ and go to 11

9. Y is an X_1

   i) Generate a uniform (0,1) random variable W_1

   ii) Set $Z \leftarrow - \ln(1.0 - W_1 * \pi_1)$

10. Choose type for Z

   i) Set $U \leftarrow U/\pi_1$

   ii) If $U \leq \alpha_1$, go to 8

   iii) Otherwise, go to 7

11. Deliver (Y,Z) and go to 4 for the FXO method, or go to 2 for the UXO and TXO methods until a sufficient number of random vectors are obtained.

Note that to compute x_ℓ and x_u in step 1 of both algorithms we use subroutine BOUND which is used the newton Raphson method to find x_ℓ and x_u.

C. REGRESSION OF Z ON Y FOR GIVEN ρ

Schmeiser (1979) has used the regression of Z on Y = y to fix the parameters in his bivariate gamma distribution. Consequently we investigate this for the mixture-truncation method case. The regression is different depending on whether $Y \leq x_0$ or $Y > x_0$. We consider two cases here, one for fixed x_0 and the other for x_0 having uniform distribution. For fixed x_0, we have

$$E[Z \mid Y = y, y \leq x_0] = \alpha_1 E[x_1] + (1-\alpha_1) E[x_2]$$
$$= 1 + x_0 - \alpha_1 x_0 \left(1 + \frac{\pi_2}{\pi_1}\right)$$

Substituting the value for α_1,

$$\alpha_1 = \pi_1 \left(1 + \frac{\rho}{x_0{}^2}\right)$$

then we have

$$E[Z \mid Y = y, y \leq x_0] = 1 + x_0 - x_0 \left(1 + \frac{\rho}{x_0{}^2}\right)$$
$$= 1 - \frac{\rho}{x_0}$$

And if $y > x_0$, then

$$E[Z \mid Y = y, y > x_0] = (1-\alpha_2) E[x_1] + \alpha_2 E[x_2]$$
$$= 1 - \frac{\pi_2}{\pi_1} x_0 + \frac{x_0}{\pi_1} \alpha_2$$

Substituting the value for α_2,

$$\alpha_2 = 1 - \frac{\pi_1}{\pi_2}(1-\alpha_1)$$

$$= 1 - \pi_1\left(1-\frac{\pi_1}{\pi_2}\frac{\rho}{x_o^2}\right)$$

then we have

$$E[Z \mid Y=y, y > x_o] = 1 + \frac{\pi_1}{\pi_2}\cdot\frac{\rho}{x_o}$$

Thus the regression is constant over $(0, x_o)$ and changes for $y \geq x_o$. This is not surprising in light of the joint distribution given in Section II-D. For uniformly distributed x_o, the computation is different for different ranges of Y. If $y \leq x_\ell$, then we have

$$E[Z \mid Y=y] = \int_{x_\ell}^{x_u} E[Z \mid Y=y, X=x_o, y \leq x_o] f(x_o) dx_o$$

$$= \int_{x_\ell}^{x_u} \left(1-\frac{\rho}{x_o^2}\right) e^{-x_o} dx_o$$

$$= -e^{-x_u} + e^{-x_\ell} - \rho \int_{x_\ell}^{x_u} \frac{e^{-x_o}}{x_o^2} dx_o$$

$$= e^{-x_\ell} - e^{-x_u} + \rho\left(\frac{1}{x_u} e^{-x_u} - \frac{1}{x_\ell} e^{-x_\ell}\right.$$

$$\left. + \ln\frac{x_u}{x_\ell} - x_u + x_\ell\right)$$

If

$$x_\ell \leq y \leq x_u$$

then we have

$$E[Z \mid Y=y] = \int_{x_L}^{y} E[Z \mid Y=y,\ X=x_0,\ y \geq x_0]\, f(x_0)\, dx_0$$

$$+ \int_{y}^{x_u} E[Z \mid Y=y,\ X=x_0,\ y \leq x_0]\, f(x_0)\, dx_0$$

$$= 2e^{-y} + 2\rho y - \rho(x_\ell + x_u) + \rho\left(\frac{1}{x_u} e^{-x_u} - \frac{1}{y} e^{-y}\right)$$

$$+ \rho \ln \frac{x_u}{y}$$

If $y > x_u$, then we have

$$E[Z \mid Y=y] = \int_{x_\ell}^{x_u} E[Z \mid Y=y,\ X=x_0,\ y > x_0]\, f(x_0)\, dx_0$$

$$= \int_{x_\ell}^{x_u} \left(1 + \frac{\pi_1}{\pi_2} \cdot \frac{\rho}{x_0}\right) e^{-x_0}\, dx_0$$

$$= e^{-x_\ell} - e^{-x_u} + \rho(y - x_\ell)$$

By making x_0 uniformly distributed over the available range of x_0 for given correlation, we can get smoother behavior for the regression function.

Ⅳ. CONCLUSION

The mixture-truncation method is a general method which can generate bivariate random vectors having any theoretical marginal distribution and allowable correlation. The generating procedure is very simple and doesn't need much computation for defining parameter values. In this respect, the mixture-truncation method is a very attractive method for generating bivariate random vectors. A price is paid for this simplicity and generality in that the Frechet bounds of correlation for the bivariate distributions specified by the marginal distribution given by Moran (1967) are not always attained. Also there is some discontinuity in the bivariate distribution. However this discontinuity can be decreased by giving some distribution to the truncation point over its range for given ρ . Thus the mixture-truncation method is very attractive for simulation studies involving only partly specified dependency structures. The mixture-truncation method may be extended to generate bivariate random vectors having negative values. Another extension may be made to use grade correlation or rank correlation which are invariant under transformation instead of using the product moment correlation.

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