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On the Equivalence of Stackelberg Strategy and Equilibrium Point in a Two-person Nonzero-sum Game

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Abstract

A sufficient condition for a Stackelberg strategy to coincide with an equilibrium point is presented. Information pattern of a Stackelberg strategy is essentially different from that of an equilibrium solution and therefore the two strategies need not be the same. However, under some restrictions on the cost functions the difference in information patterns between the two strategies can be disregarded so that the two strategies coincide. The result is extended to the case of discrete-time dynamic games.

I. INTRODUCTION

In a two-person zero-sum game, the objectives of the two players are exactly opposite and the optimal strategy (u^0, v^0) will satisfy the saddle point condition

$$J(u^0, v) \leqslant J(u^0, v^0) \leqslant J(u, v^0) \quad \forall u, v, \tag{1}$$

where $J(u,v)$ is the payoff function which depends on the minimizing player's strategy u and the maximizing player's strategy v. In two-person nonzero-sum games, on the other hand, the objectives of the players are neither exactly opposite nor do they coincide with each other. Thus, there are several ways of defining an "optimal strategy" according to the rationality assumed by each player. Corresponding to the saddle point condition of a zero-sum game, there is an equilibrium point condition for a nonzero-sum game. If $J_1(u,v)$ and $J_2(u,v)$ are cost functions for Players 1 and 2 and each player's goal is to minimize his own cost function, then an equilibrium point (u^*, v^*) will satisfy the following conditions:

$$J_1(u^*, v^*) < J_1(u, v^*) \quad \forall u \tag{2}$$

$$J_2(u^*, v^*) < J_2(u^*, v) \quad \forall v \tag{3}$$

A player who selects an equilibrium strategy is assured that this course of action will minimize his cost function provided his opponent holds fast to his equilibrium strategy.

In this paper a condition for the existence of equilibrum point is derived using the strategy

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suggested by von Stackelberg (discussed in [1], [2], and [7]). Originally the Stackelberg strategy is defined for the case where the information pattern is biased in the sense that the first player does not know the cost function of the second, but the second player knows both of the cost functions. In the equilibrium conditions, on the other hand, there are no restrictions on the information pattern of the cost function. But if we place some restrictions on the cost functions, we can disregard the difference in information pattern between the two strategies and derive conditions for which the Stackelberg strategy is the same as the equilibrium solution.

II. A SUFFICIENT CONDITION

DEFINITION 1 : Given a two-person game where Player 1 wishes to minimize cost function $J_1(u,v)$ and Player 2 wishes to minimize cost function $J_2(u,v)$ by choosing u,v from admissible strategy sets U and V, respectively, the strategy set $(u_2{}^0,v_2{}^0)$ is called a Stackelberg strategy with Player 2 as leader and Player 1 as follower if for any $u\in U$ and $v\in V$,

$$J_2(u_2{}^0,v_2{}^0)\leqslant J_2(\bar{u}(v),v), \tag{4}$$

where

$$J_1(\bar{u}(v),v)=\min_{u\in U} J_1(u,v) \tag{5}$$

and

$$u_2{}^0=\bar{u}(v_2{}^0). \tag{6}$$

The definition of a Stackelberg strategy with Player 1 as leader and Player 2 as follower would parallel the above definition: $(u_1{}^0,v_1{}^0)$ is a Stackelberg strategy with Player 1 as leader if for any $u\in U$ and $v\in V$,

$$J_1(u_1{}^0,v_1{}^0)\leqslant J_1(u,\bar{v}(u)), \tag{7}$$

where

$$J_2(u,\bar{v}(u))=\min_{v\in V} J_2(u,v) \tag{8}$$

and

$$v_1{}^0=\bar{v}(u_1{}^0). \tag{9}$$

A Stackelberg strategy with Player 2 as leader is the optimal strategy for Player 2 if Player 2 announces his move first and if the goal of Player 1 is to minimize J_1, while that of Player 2 is to minimize J_2. By announcing his Stackelberg strategy $v_2{}^0$ first, Player 2 forces Player 1 to follow and use the Stackelberg strategy $u_2{}^0$ since the follower will do no better than to follow a Stackelberg strategy himself.

In a two-person game, the value of the cost function depends on the order in which the players choose their strategies. In zero-sum games, the player who chooses second can either improve the value of his cost function or be at least assured of not being worse off than if he had to choose first. In nonzero-sum games, on the other hand, the player who chooses first can force the game to the solution point which produces the smallest value of his cost criterion. Thus, the cost for the leader of the Stackelberg strategy is less than or equal to his cost when the equilibrium strategy is used, that is,

$$J_1(u_1{}^0,v_1{}^0)\leqslant J_1(u^*,v^*) \tag{10}$$

$$J_2(u_2^0, v_2^0) \leqslant J_2(u^*, v^*) \tag{11}$$

In this section, a condition for equality in the above two inequalities is derived. Many authors (3, 4, and 5) have proved the following existence theorem of an equilibrium point.

THEOREM 2 : Let U and V be two compact convex sets in R_m and R_n, respectively. Let $J_1(u,v)$ and $J_2(u,v)$ be two continuous functions on $U\times V$, and further let $J_1(u,v)$ be convex in u for fixed v and $J_2(u,v)$ convex in v for fixed u. Then, there exists an equilibrium pair (u^*,v^*).

Let

$$X_1=\left\{\tilde{u} : J_1(\tilde{u},\bar{v}(\tilde{u}))=\min_{u\in U} J_1(u,\bar{v}(u))\right\}, \tag{12}$$

$$Y_1(u)=\left\{\bar{v}(u) : J_2(u,\bar{v}(u))=\min_{v\in V} J_2(u,v)\right\}, \tag{13}$$

$$X_2(v)=\left\{\bar{u}(v) : J_1(\bar{u}(v),v)=\min_{u\in U} J_1(u,v))\right\}, \tag{14}$$

$$Y_2=\left\{\tilde{v} : J_2(\bar{u}(\tilde{v}),\tilde{v})=\min_{v\in V} J_2(\bar{u}(v),v)\right\}. \tag{15}$$

If there are no restrictions placed on the cost functions $J_1(u,v)$, and $J_2(u,v)$, then there may not be a unique strategy pair which minimizes these functions. Although the first player is indifferent to whichever strategy, in $X_2(v)$, he selects since they all produce the same value of his payoff, $J_1(u,v)$, different strategies may have a large influence on the second player's payoff, $J_2(u,v)$. Thus the sets $X_1, X_2(v)$, $Y_1(u)$, and Y_2 may not be singleton sets.

In the Stackelberg solution, it is assumed that the follower does not know the leader's cost function. So the nonuniqueness of the follower's strategy, $\bar{u}(v)$, is irrelevant in the derivation of Stackelberg solution. Thus, we must make some restrictions on the nonuniqueness of the strategy, $\bar{u}(v)$, to relate the Stackelberg solution to the equilibrium point.

The following theorem gives sufficient conditions for a Stackelberg strategy to coincide with an equilibrium point.

THEOREM 3 : Let U and V be two compact convex sets in R^m and R^n, respectively. Let $J_1(u,v)$ and $J_2(u,v)$ be two continuous functions on $U\times V$, and further let $J_1(u,v)$ be convex in u for fixed v and $J_2(u,v)$ convex in v for fixed u. Then we have the following:

(Ⅰ) if X_2 and Y_2 are singleton sets, and $J_2(\bar{u}(v),v)\leqslant J_2(u_2^0,v)$, then the Stackelberg strategy with Player 2 as leader, (u_2^0,v_2^0), is also an equilibrium solution.

(Ⅱ) If X_1 and Y_1 are singleton sets, and $J_1(u,\bar{v}(u))\leqslant J_1(u,v_1^0)$, then the Stackelberg strategy with Player 1 as leader, (u_1^0,v_1^0), is also an equilibrium solution.

PROOF: From equation (5), $J_1(\bar{u}(v),v)=\min_{u\in U} J_1(u,v)$.

In general $\min_{u\in U} J_1(u,v)\leqslant J_1(u,v)\ \forall u\in U$ and $v\in V$.

Thus, we have $J_1(\bar{u}(v),v)\leqslant J_1(u,v)$.

Letting $v=v_2^0$ in the above equation and in view of equation (6) we have $J_1(u_2^0,v_2^0)\leqslant J_1(u,v_2^0)$.

On the other hand, from equation (4), $J_2(u_2^0,v_2^0)\leqslant J_2(\bar{u}(v),v)$.

By hypothesis $J_2(\bar{u}(v),v)\leqslant J_2(u_2^0,v)$.

Thus, we have $J_2(u_2^0,v_2^0)\leqslant J_2(u_2^0,v)$.

Hence (u_2^0,v_2^0) satisfies both conditions (equation(2) and (3)) for an equilibrium point.

Part (Ⅱ) can be similarly handled.

The following corollary is a direct consequence of Theorem 3.

COROLLARY 4 : If either condition (Ⅰ) or (Ⅱ) of Theorem 3 is satisfied and the two Stackelberg are identical then both Stackelberg solutions are the same as an equilibrium point, that is,

$$u_1^0=u_2^0=u^* \tag{16}$$

and

$$v_1^0=v_2^0=v^*. \tag{17}$$

We shall illustrate the foregoing discussions by a simple numerical example.

EXAMPLE 1 : Consider the case where Players 1 and 2 wish to minimize their cost functions J_1 and J_2, respectively, where

$$J_1(u,v)=(2u+v+4)^2,$$

$$J_2(u,v)=(u+2v-1)^2, \quad |u|\leqslant 5, \quad |v|\leqslant 5.$$

(Ⅰ) From equation (5), $\bar{u}(v)=-\frac{1}{2}v-2, \ |v|\leqslant 5,$

Player 2's payoff then becomes $J_2(\bar{u}(v),v)=\left(\frac{3}{2}v-3\right)^2$ whose minimum occurs at $v=v_2^0=2$.

The corresponding strategy for Player 1 becomes $u_2^0=\bar{u}(v_2^0)=-3$.

Thus, the Stackelberg solution with Player 2 as leader is given by $(u_2^0,v_2^0)=(-3,2)$ with $J_1(u_2^0,v_2^0)=J_2(u_2^0,v_2^0)=0$.

(Ⅱ) From equation (8), $\bar{v}(u)=-\frac{1}{2}u+\frac{1}{2}, \ |u|\leqslant 5.$

Player 1's payoff then becomes $J_1(u,\bar{v}(u))=\left(\frac{3}{2}u+\frac{9}{2}\right)^2$ whose minimum occurs at $u=u_1^0=-3$.

The corresponding strategy for Player 2 becomes $v_1^0=\bar{v}(u_1^0)=2$.

Thus, the Stackelberg solution with Player 1 as leader is given by $(u_1^0,v_1^0)=(-3,2)$ with $J_1(u_1^0,v_1^0)=J_2(u_1^0,v_1^0)=0$.

Observe that all conditions of Theorem 3 are satisfied in this game: Both J_1 and J_2 are continuous and J_1 is convex in u for all v and J_2 is convex in v for all u. Both $\bar{u}(v)$ and $\bar{v}(u)$ are unique. Finally, $J_1(u,\bar{v}(u))=\left(\frac{3}{2}u+\frac{9}{2}\right)^2 \leqslant J_1(u,v_1^0)=(2u+6)^2$.

Thus, from Theorem 3 and Corollary 4, the Stackelberg solution point $(-3,2)$ is also an equilibrium point.

Ⅲ. DISCRETE MULTISTAGE GAMES

Theorem 3 can also be applied to discrete multistage games.

Let the transition of stages at stage i be described by

$$z(i+1)=g_i\,[z(i),u(i),v(i)], \quad i=1,\cdots, N, \tag{18}$$

where

$z(i)$ = game state at i-th stage,

$u(i)$ = Player 1's strategy at the i-th stage,

$v(i)$ = Player 2's strategy at the i-th stage.

The objective functions for Players 1 and 2 are

$$J_1=\sum_{i=1}^{N} L_1\,[i,z(i),u(i),v(i)], \tag{19}$$

$$J_2=\sum_{i=1}^{N} L_{2i}[i,z(i),u(i),v(i)], \tag{20}$$

where $L_j[i,z(i),u(i),v(i)]$ is the payoff function of Player j at the i-th stage game. Player 1 wishes to choose $u(1),\cdots, u(N)$ so as to minimize J_1, while Player 2 wishes to pick $v(1),\cdots, v(N)$ so as to minimize J_2.

The following definition of feedback Stackelberg strategy is analogous to the one given by Chen and Cruz [1].

DEFINITION 5 : A feedback Stackelberg strategy with Player 2 as leader is defined as $[u_2{}^0(i), v_2{}^0(i)], i=1,\cdots, N$, satisfying

$$W_1(i)=K_{1i}[i,z(i),u_2{}^0(i),v_2{}^0(i)], \tag{21}$$

and

$$W_2(i)=\min_{v(i)\in V_i} K_2[i,z(i),\bar{u}(i,v(i)),v(i)]. \tag{22}$$

where

$$K_j[i,z(i),u(i),v(i)]=W_j(i+1)+L_j[i,z(i),u(i),v(i)],\quad j=1,2, \tag{23}$$

$$K_1[i,z(i),\bar{u}(i,v(i)),v(i)]=\min_{u(i)\in U_i} K_1[i,z(i),u(i),v(i)], \tag{24}$$

$$u_2{}^0(i)=\bar{u}(i,v_2{}^0(i)), \tag{25}$$

$W(i+1)$ is the value of the objective function of Player 1 for the last $(N-i)$ stages when feedback Stackelberg strategy with Player 2 as leader is used, $W(N+1)=0$, and U_i and V_i are the sets of all admissible strategies of Players 1 and 2, respectively, at the i-th stage game.

The feedback Stackelberg strategy with Player 1 as leader, $[u_1{}^0(i),v_1{}^0(i)]$, $i=1,\cdots,N$, can be defined similarly.

Let

$$X_1(i)=\Big\{\tilde{u}(i) : K_1[i,z(i),\tilde{u}(i),\bar{v}(i,\tilde{u}(i))] =\min_{u(i)\in U_i} K_1[i,z(i),u(i),\bar{v}(i,u(i))]\Big\}, \tag{26}$$

$$Y_1(i,u(i))=\Big\{\bar{v}(i,u(i)) : K_2[i,z(i),u(i),\bar{v}(i,u(i))] =\min_{v(i)\in V_i} K_2[i,z(i),u(i),v(i)]\Big\}, \tag{27}$$

$$X_2(i,v(i))=\Big\{\bar{u}(i,v(i)) : K_1[i,z(i),\bar{u}(i,v(i)),v(i)] =\min_{u(i)\in U_i} K_1[i,z(i),u(i),v(i)]\Big\}, \tag{28}$$

and

$$Y_2(i)=\Big\{\tilde{v}(i) : K_2[i,z(i),\bar{u}(i,\tilde{v}(i)),\tilde{v}(i)] =\min_{v(i)\in V_i} K_2[i,z(i),\bar{u}(i,v(i)),v(i)]\Big\}. \tag{29}$$

Then, sufficient conditions for a feedback Stackelberg strategy to be a feedback equilibrium solution [6] for a discrete multistage game are given by the following theorem.

THEOREM 6 : Let U_i and V_i, $i=1,\cdots, N$, be compact convex sets in R^m and R^n. Let $K_j[i,z$

$(i), u(i), v(i)]$ be continuous functions defined on $U_i \times V_i$ for all $i=1, \cdots, N$, and $J=1, 2$, and suppose K_1 is convex in $u(i)$ for fixed $v(i)$ and K_2 is convex in $v(i)$ for fixed $u(i)$ for all $i=1, \cdots, N$. Then we have the following:

(Ⅰ) If $X_2(i, v(i))$ and $Y_2(i)$ are singleton sets and

$K_2[i, z(i), \bar{u}(i, v(i)), v(i)] \leqslant K_2[i, z(i), \bar{u}(i, v_2^0(i)), v(i)]$ for all $i=1, \cdots, N$, then the feedback Strackelberg strategies with Player 2 as leader, $[u_2^0(i), v_2^0(i)]$, $i=1, \cdots, N$, are also feedback equilibrium solutions.

(Ⅱ) If $X_1(i)$ and $Y_1(i, u(i))$ are singleton sets and

$K_1[i, z(i), u(i), \bar{v}(i, u(i))] \leqslant K_1[i, z(i), u(i), \bar{v}(i, u_1^0(i))]$ for all $i=1, \cdots, N$, then the feedback Stackelberg strategies with Player 1 as leader, $[u_1^0(i), v_1^0(i)]$, $i=1, \cdots, N$, are also feedback equilibrium solutions.

PROOF : The proof follows the same lines as Theorem 3.

COROLLARY 7 : If either condition (Ⅰ) or (Ⅱ) of Theorem 6 is satisfied and if $[u_1^0(i), v_1^0(i)]$ is equal to $[u_2^0(i), v_2^0(i)]$ for all $i=1, \cdots, N$, then

$$u_1^0(i) = u_2^0(i) = u^*(i), \; i=1, \cdots, N, \tag{30}$$

and

$$v_1^0(i) = v_2^0(i) = v^*(i), \; i=1, \cdots, N, \tag{31}$$

where $[u^*(i), v^*(i)]$ is the feedback equilibrium solution at the i-th stage game.

PROOF : The proof follows directly from Theorem 6.

We shall illustrate the foregoing discussions by a simple multistage game.

EXAMPLE 2 : Consider the following 3-stage linear system with quadratic payoff functions. The evolution of the state of the system is determined by linear difference equations

$$z(i+1) = z(i) + u(i) + v(i), \; i=1, 2, 3, \; z(1) = 3.$$

The cost functions for Players 1 and 2 are

$$J_1 = \sum_{i=1}^{3} a_i z(i+1)^2 + b_i u(i)^2 + c_i v(i)^2$$

and

$$J_2 = \sum_{i=1}^{3} d_i z(i+1)^2 + e_i u(i)^2 + f_i v(i)^2,$$

respectively, where the parameters of the system a_i, b_i, c_i, d_i, e_i, and f_i are assumed to have the following values.

Parameters / Stages (i)	a_i	b_i	c_i	d_i	e_i	f_i
1	3	4	4	2	3	3
2	3	4	2	3	4	2
3	4	2	4	4	2	4

Each player wishes to minimize his cost function and all system parameters are assumed to be known to both players. Both $u(i)$ and $v(i)$ are assumed to belong to sets U_i and V_i where

$$U_i = \{u(i) : |u(i)| \leqslant 5\}, \quad i = 1, 2, 3,$$
$$V_i = \{v(i) : |v(i)| \leqslant 5\}, \quad i = 1, 2, 3.$$

This game can be solved using the method of Example 1 and dynamic programming. The following table summarizes the feedback Stackelberg solutions with Player 2 as leader for each stage game.

Stages(i)	$[u_2^0(i),\ v_2^0(i)]$	$W_1(i)$	$W_2(i)$
1	$[-1.0,\ -1.0]$	12	9
2	$\left[-\frac{1}{4},\ -\frac{1}{2}\right]$	1.0	1.0
3	$\left[-\frac{1}{8},\ -\frac{1}{16}\right]$	$\frac{1}{16}$	$\frac{1}{16}$

It is easy to verify that all conditions of Theorem 6 are satisfied. Thus, the feedback Stackelberg solutions are also feedback equilibrium points. These solutions can be shown to satisfy both conditions of equations (2) and (3).

Ⅳ. CONCLUSION

In this paper, sufficient conditions for a Stackelberg strategy to coincide with an equilibrium point are presented. In general it is simpler to derive a Stackelberg strategy than an equilibrium point. Thus, if the cost functions satisfy these sufficient conditions, the equilibrium point can be derived using the Stackelberg solution approach. The discussions on the relationship between equilibrium points and Stackelberg strategies will apply equally well to differential games.

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