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ON STATIONARY GAUSSIAN SECOND ORDER MARKOV PROCESSES

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0. Abstract

In this paper we give a characterization of Stationary Gaussian 2nd order Markov processes in terms of its covariance function $R(\tau)=E[X(t)X(t+\tau)]$ and also give some relationship among quasi-Markov, Markov and 2nd order Markov processes.

1. Introduction

Let $\{X(t): t\in T\}$ be a stationary Gaussian process with a covariance function $R(\tau)=E[X(t)X(t+\tau)]$. Assume that $E(X(t))=0$ and $E(X(t))^2=1$ for each $t\in T$. A process $X(t)$ is called *Markov* if for any partition of T with $t_1<t_2<\cdots<t_k<t$,

$$P\{X(t)\leq x\,|\,X(t_1),\ X(t_2),\ \cdots,\ X(t_k)\}=P\{X(t)\leq x\,|\,X(t_k)\}$$

and is called *2nd order Markov* if for any partition of T with $t_1<t_2<\cdots<t_{k-1}<t_k<t$,

$$\begin{aligned}P\{X(t)\leq x\,|\,X(t_1),\ X(t_2),\ \cdots,\ X(t_{k-1}),\ X(t_k)\}\\=P\{X(t)\leq x\,|\,X(t_{k-1}),\ X(t_k)\}\end{aligned}$$

for any real number x. One can define the concept of nth order Markov in similar fashion. A process $X(t)$, $t\in T$, is called *quasi-Markov* (or *reciprocal*) if for any $A\in\sigma(X(s),\ s<t \text{ or } s>u)$, $B\in\sigma(X(s),\ t<s<u)$ and for each pair t and u with $t<u$ in T,

$$P\{A\cap B\,|\,X(t),\ X(u)\}=P\{A\,|\,X(t),\ X(u)\}\cdot P\{B\,|\,X(t),\ X(u)\}.$$

It is well known that the stationary Gaussian process is Markov (with continuous covariance function) if and only if its covariance function has the following form:

$$R(\tau)=e^{-\alpha|\tau|}\quad\text{where } \alpha\geq 0$$

(See Doob [2] for $T=[0,\infty)$). Chay [1] and Jamison [3] have shown recently that the stationary Gaussian process is quasi-Markov if and only if its covariance function has one of the following forms:

(i) $R(\tau)=e^{-\alpha|\tau|}$ $(\alpha\geq 0)$,

(ii) $R(\tau)=\cos\beta\tau$, or

(iii) $R(\tau)=1-|\tau|$, $\tau\in(-1,1)$

In this paper we characterized the 2nd order Markov (Stationary Gaussian) process. Our main result is as follows: A process $X(t)$, $t\in T=[0,\zeta]$ is 2nd order Markov if and only if its covariance function has one of the following forms:

(i) $R(\tau)\equiv 1$

(ii) $R(\tau)=\lambda^{\tau}$ where $\lambda<1$ or

(iii) $R(\tau)=\cos\gamma\tau$ where $\gamma\tau<\pi$

We have also shown that

(i) when T is a set of integers, 2nd order Markov property and quasi-Markov property are equivalent,

(ii) when T is a finite interval of real numbers, 2nd order Markov property implies quasi-Markov property, and

(iii) when T is $[0,\infty)$, 2nd order Markov property implies Markov property.

2. Characterization of 2nd order markov

Denote $R^+=[0,\infty)$,

$Z^+=$the set of all positive integers

$A=[0,\zeta]$ for finite real number ζ.

We assume that the process $X(t)$, $t\in T$, (where T may be R^+, Z^+ or A) be stationary and Gaussian with $E(X(t))=0$ and $E(X(t))^2=1$ for each $t\in T$. Let $R(\tau)$ be the covariance function of the process $X(t)$.

PROPOSITION 1. *Let $X(t)$, $t\in R^+$ (or Z^+) be nth order Markov process such that $|R(t_0)|=1$ for some $t_0\neq 0$ in R^+, then $|R(s)|=1$ for $\forall$ $s\in R^+$.*

PROOF. Assume that $R(t_0)=1$ (the case when $R(t_0)=-1$ can be proved by a similar argument). Since the process is Gaussian, $R(t_0)=1$ implies that $X(0)$ and $X(t_0)$ are linearly dependent random variables, and we have $X(0)=X(t_0)$a. s. because $R(t_0)=1$. The stationarity of the porcess implies that

(1) $$X(0)=X(t_0)=X(kt_0) \text{ for } k\in Z^+.$$

For $s\in(nt_0, (n+1)t_0)$, let $E\{X(s)|X(0)\}=aX(0)$. Then $X(s)-aX(0)\perp X(0)$ implies that $R(s)=a$. Now for $u\in(0, t_0)$,

$$E\{X(s)\mid X(u),\ X(t_0),\ \cdots,\ X(nt_0)\}$$
$$=E\{X(s)\mid X(t_0),\ \cdots,\ X(nt_0)\}=E\{X(s)\mid X(0)\}$$
$$=aX(0),\ \text{which implies that } X(s)-aX(0)\perp X(u).$$

Therefore, $E[\{X(s)-aX(0)\}\cdot X(u)]=0$, i.e.

$$E[X(s)X(u)]=aE[X(0)X(u)]=aR(u)=R(s)R(u),\ \text{hence}$$

(2) $\quad R(s-u)=R(s)R(u)$ for $u\in(0,\ t_0)$ and $s\in(nt_0,\ (n+1)t_0)$.

Let $s=nt_0+v$, then we have $v\in(0,\ t_0)$ and

(3) $\quad R(nt_0+v)=E[X(nt_0+v)X(0)]=E[X(nt_0+v)X(nt_0)]$
$$=E[X(v)X(0)]=R(v).$$

Hence from (2) and (3) $R(v-u)=R(v)R(u)$ for $u,\ v$ in $(0,\ t_0)$ and by letting $u=v$, we have $R(0)=R^2(u)$, i.e. $|R(u)|=1$ for $\forall\ u\in(0,\ t_0)$. Now the result that $|R(u)|=1$ for $\forall\ u\in R^+$ follows from the stationarity of the process $X(t)$. This completes the proof.

THEOREM 1. *A process $X(t)$, $t\in Z^+$ is 2nd order Markov if and only if its covariance function $R(\tau)$ has one of the following forms*;

(i) $|R(\tau)|\equiv 1$

(ii) $R(\tau)=\lambda^\tau$, *where* $\lambda=R(1)$

(iii) $R(\tau)=\cos\gamma\tau$, *where* $R(1)=\cos\gamma$ *and* π/γ *is irrational number.*

PROOF. Assume that $|R(t)|\neq 1$ for $\forall\ t\in Z^+$. By the properties of Gaussian and 2nd order Markov, we obtained, for $0\leq u<v<w<s$ in Z^+,

$$E\{X(s)\mid X(u),\ X(v),\ X(w)\}=E\{X(s)\mid X(v),\ X(w)\}=aX(w)+bX(v).$$

Note that the linear expression $aX(w)+bX(u)$ follows from the Gaussian property. Thus $X(s)-aX(w)-bX(v)\perp X(u)$, $X(v)$ and $X(w)$, which implies:

(4) $\quad R(s-u)-aR(w-u)-bR(v-u)=0$
$$R(s-v)-aR(w-v)-b=0$$
$$R(s-w)-a-bR(w-v)=0.$$

Let $v-u=l_1$, $w-v=l_2$ and $s-w=l_3$, then equations in (4) are equivalent to the following equations:

(5) $\quad R(l_1+l_2+l_3)-aR(l_1+l_2)-bR(l_1)=0$

(6) $\quad R(l_2+l_3)-aR(l_2)-b=0$

(7) $\quad R(l_3)-a-bR(l_2)=0.$

Since $R(t)\neq 1\ \forall\ t\in Z^+$, we can solve (6) and (7) and obtain

(8) $\quad b=[R(l_2+l_3)-R(l_2)R(l_3)]/[1-R^2(l_2)]$

$$a=[R(l_3)-R(l_2)R(l_2+l_3)]/[1-R^2(l_2)].$$

Substituting (8) into (5), we have

$$(9) \qquad [1-R^2(l_2)]R(l_1+l_2+l_3)=R(l_1)[R(l_2+l_3)-R(l_2)R(l_3)] + R(l_1+l_2)[R(l_3)-R(l_2)R(l_2+l_3)].$$

Set $l_2=l_3=1$ and $l_1=l$, then (9) becomes

$$(10) \qquad (1-k_1^2)R(l+2)-k_1(1-k_2)R(l+1)-(k_2-k_1^2)R(l)=0$$

where $k_1=R(1)<1$ and $k_2=R(2)<1$.

To solve the linear difference equation (10), we suppose that λ_1 and λ_2 are the roots of $(1-k_1^2)\chi^2-k_1(1-k_2)\chi-(k_2-k_1^2)=0$, then the solutions of the equation (10) would be:

(i) $(A_1+B_1l)\lambda^l$ if $\lambda_1=\lambda_2=\lambda$

(ii) $A_2\lambda_1^l+B_2\lambda_2^l$ if $\lambda_1\neq\lambda_2$ and both are real

(iii) $A_3 \cos \gamma l+B_3 \sin \gamma l$ if λ_1, λ_2 are complex,

for some constant A_i and B_i, $i=1, 2, 3$.

The only forms from (i), (ii) and (iii), which satisfy the non-linear difference equation (9), are as following:

$$(11) \qquad R(\tau)=\lambda^\tau \text{ and } R(\tau)=\cos \gamma\tau.$$

We note that if $R(\tau)=\cos \gamma\tau$, then $|R(t)|\neq 1$ for $\forall\ t\in Z^+$ (by Proposition 1) and therefore $\frac{\pi}{\gamma}$ has to be irrational. This completes the proof of "only if" part of the theorem. This completes the proof since the proof of "if" part is trival.

THEOREM 2. *Assume that the continuous covariance function $R(\tau)$ is differentiable for at least one point in $(0, \zeta)$ and it has the right derivative at $\tau=0$. The process $X(t)$, $t\in A=[0, \zeta]$ is 2nd order Markov iff $R(\tau)$ has one of the following forms:*

(i) $R(\tau)\equiv 1$

(ii) $R(\tau)=\lambda^\tau$, *where* $\lambda<1$

(iii) $R(\tau)=\cos \gamma\tau$ *where* $\gamma\ \zeta<\pi$

PROOF. Assume that the process is 2nd order Markov. Suppose that there exists $t_0\in A\setminus\{0\}$ such that $|R(t_0)|=1$, then we want to show that $R(\tau)\equiv 1$ for all $\tau\in A$. Letting $l_2=0$ in (5), (6) and (7), we obtained:

(i) when $R(l_2)=1$, $\begin{cases} R(l_1+l_3)=aR(l_1)+bR(l_1) \\ R(l_3)=a+b \end{cases}$

i.e. $R(l_1+l_3)=R(l_1)R(l_3)$ for $\forall$ l_1, l_3 in $A\backslash\{0\}$,

therefore $R(\tau)=\lambda^\tau$ for $\forall\tau\in A$. But $R(t_0)=1$ implies that $\lambda=1$, and hence $R(t)\equiv 1$.

(ii) when $R(l_2)=-1$, $\begin{cases} R(l_1+l_3)=aR(l_1)-bR(l_1) \\ R(l_3)=a-b \end{cases}$

i.e. $R(l_1+l_3)=R(l_1)\cdot R(l_3)$ for $\forall$ l_1, l_3 in $A\backslash\{0\}$,

therefore $R(\tau)=\lambda^\tau$ for $\forall$ $\tau\in A$. But $R(t_0)=-1$ implies that $\lambda^{t_0}=-1$ i.e. $\lambda=-1$, which is impossible since $R(\tau)$ is continuous.

Now we assume that $|R(\tau)|\neq 1$ for $\forall$ $\tau\in A\backslash\{0\}$, then $R(\tau)$ should satisfy the equation (9) for any l_1, l_2, l_3 in A. i.e.

$$\begin{aligned} (12) \qquad & (1-R^2(l_2))\,[R(l_1+l_2+l_3)-R(l_2+l_3)] \\ & =(R(l_1)-R(0))\,[R(l_2+l_3)-R(l_2)R(l_3)] \\ & +[R(l_1+l_2)-R(l_2)]\,[R(l_3)-R(l_2)R(l_2+l_3)] \end{aligned}$$

Now dividing the above equation by l_1 and taking the limit as l_1 approaches zero, we obtained

$$\begin{aligned} R'(0)\,[R(l_2+l_3)-R(l_2)R(l_3))= & \lim_{l_1\to 0}\Big[(1-R^2(l_2))\frac{R(l_1+l_2+l_3)-R(l_2+l_3)}{l_1} \\ (13) \qquad & -(R(l_3)-R(l_2)R(l_2+l_3))\frac{R(l_1+l_2)-R(l_2)}{l_1}\Big] \end{aligned}$$

Since $R(\tau)$ is differentiable at one point and l_2, l_3 are arbitrary, we conclude that $R(\tau)$ is differentiable everywhere in $(0, \zeta)$.

Rewrite (9) as:

$$\begin{aligned} (14) \qquad & R(l_1+l_2+l_3)(1+R(l_2))(1-R(l_2))=R(l_1)\cdot \\ & [R(l_2+l_3)-R(l_3)-R(l_3)(R(l_2)-R(0))]+R(l_1+l_2)\,[R(l_3)-R(l_2+l_3) \\ & -R(l_2+l_3)(R(l_2)-R(0))]. \end{aligned}$$

Dividing (14) by l_2 and taking the limit as l_2 approaches zero, we obtained

$$2R'(0)\,[R(l_1)R(l_3)-R(l_1+l_3)]=0$$

Thus, we have either $R'(0)=0$ or $R(l_1)R(l_3)=R(l_1+l_3)$.

(Case 1) If $R(l_1)R(l_3)=R(l_1+l_3)$, then general solution is $R(\tau)=A_1\lambda^\tau$. Thus we have $R(\tau)=\lambda^\tau$ for $\forall$ $\tau\in A$ since the solution has to satisfy the initial condition $R(0)=1$.

(Case 2) If $R'(0)=0$, from (13) we have

$$(1-R^2(l_2))R'(l_2+l_3)=R'(0)[R(l_2+l_3)-R(l_2)R(l_3)]$$
$$+R'(l_2)[R(l_3)-R(l_2)R(l_2+l_3)], \text{ i.e.}$$

(15) $$[1-R^2(l_2)][R'(l_2+l_3)-R'(l_2)]=R'(0)[R(l_2+l_3)-R(l_2)$$
$$-R(l_2)(R(l_3)-R(0)]+R'(l_2)[R(l_3)-R(0)-R(l_2)(R(l_2+l_3)-R(l_2)].$$

Now dividing (15) by l_3 and taking the limit as l_3 approaches zero, we can show that $R''(\tau)$ exists for $\forall\tau\in(0,\zeta)$. Also we obtained

(16) $$(1-R^2(l_2))R''(l_2)=2R'(0)R'(l_2)-(R'(0))^2R(l_2)-R(l_2)(R'(l_2))^2$$

Since $R'(0)=0$, we have

(17) $$(1-y^2)y''=-y(y')^2 \text{ where } y=R(\tau).$$

To solve (17), let $p=y'$, then (17) becomes $(1-y^2)\,p\frac{dp}{dy}=-yp^2$, i.e. either $p=0$ or $(1-y^2)\,\frac{dp}{dy}=-yp$. If $p\equiv0$, then $R(\tau)\equiv1$. If $(1-y^2)\,\frac{dp}{dy}=-yp$, then $\frac{dp}{p}=-\frac{ydy}{1-y^2}$ (since $y\neq1$), i.e. $\ln|p|=\frac{1}{2}\ln(1-y^2)+c''$ or $|p|=c'\sqrt{1-y^2}$ or $\frac{dy}{dx}=c'\sqrt{1-y^2}$. Therefore $y=R(\tau)=\cos\lambda\tau$.

We note that $\gamma\zeta<\pi$ since $R(\tau)\neq1$. This completes the proof of "only if" part of the theorem. Since the proof of "if" part is trivial, this completes the proof.

The following Corollaries can be easily obtained by comparing the results of Chay [1] and ours.

COROLLARY 1. *A process $X(t)$, $t\in Z^+$ in 2nd order Markov iff it is quasi-Markov, where Z^+ is the set of all positive integers.*

COROLLARY 2. *If a process $X(t)$, $t\in A$ is 2nd order Markov, then it is also quasi-Markov, where $A=[0,\xi]$ for a finite real number ξ.*

COROLLARY 3. *If a process $X(t)$, $t\in R^+$ is 2nd order Markov, then it is a Markov process, where $R^+=[0,\infty)$.*

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