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A RESULT ON FIXED POINTS

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1. Introduction

In recent years many extensions and generalizations of Banach fixed point theorem had been done by many authors. But in all the cases the mappings under consideration involve only two points of the space. Until, recently Pittnauer [4] and also Rhoades [6] studied contractive type mappings involving three points of the space. Pittnauer [5] also studied the mapping involving four points of the space.

The aim of this paper is to establish a fixed point theorem containing four points of the space and we shall show that results of Banach [1], Browder [2], Kannan [3], Pittnauer [5] and Reich [7] as special cases. Let (x, d) be a closed bounded subset of a complete metric space. Let $\Psi_i (i=1, 2, 3)$ be monotone increasing selfmapping of the reals $t \geq 0$, which is continuous on the right and satisfy the condition

(1.1) $\underset{i=1,2,3}{\Psi_i(t)} < \frac{t}{3}$ for $t>0$ and $\underset{i=1,2,3}{\Psi_i(0)}=0$ for $t=0$.

Also, let f be a mapping of X into itself such that

(1.2) $d(fu_1, fu_2) \leq \Psi_1[d(u_1, u_2)] + \Psi_2[d(u_1, f^k u_3)] + \Psi_3[d(u_2, f^l u_4)]$

for $u_1, u_2, u_3, u_4 \in X$ and fixed integers $0 \leq k \leq l$.

2. Fixed point theorem

The following theorem is patterned after the results of Pittnauer [5] with necessary modifications as required for the more general settings.

THEOREM 2.1. *If f be a selfmapping of X into itself satisfying condition (1.2), then f has a unique fixed point.*

PROOF. Let $x, y \in X$ and we define

$$u_1 = f^l x, \quad u_2 = f^k y, \quad u_3 = y, \quad u_4 = x.$$

Then the condition (1.2) takes the form

(2.1) $d(f^{l+1}x,\ f^{k+1}y)\leq\Psi_1[d(f^lx,\ f^ky)]+\Psi_2[d(f^lx,\ f^ky)]+\Psi_3[d(f^lx,\ f^ky)]$.

We now choose an arbitrary $x_0\in X$ and define $x=f^{n+k}x_0$, $y=f^{n+l+m}x_0$ for some fixed positive integer n, m, then from (2.1) we get

(2.2) $$d(f^{n+k+l+m+1}x_0,\ f^{n+k+l+1}x_0)\leq\Psi_1[d(f^{n+k+l+m}x_0,\ f^{n+k+l}x_0)] +\Psi_2[d(f^{n+k+l+m}x_0,\ f^{n+k+l}x_0)]+\Psi_3[d(f^{n+k+l+m}x_0,\ f^{n+k+l}x_0)].$$

We shall show that the iterated sequence $I(f,\ x_0)=\{x_n|x_n=f^nx_0,\ n=0,\ 1,\ 2\cdots\}$ is a Cauchy sequence. For $n\geq k+l$, the inequality (2.2) implies

(2.3) $$d(x_{n+m+1},\ x_{n+1})\leq\Psi_1[d(x_{n+m},\ x_n)]+\Psi_2[d(x_{n+m},\ x_n)]+\Psi_3[d(x_{n+m},\ x_n)].$$

Let $\beta_n=\sup_{m\geq 0} d(x_n,\ x_{n+m})$ for $n\geq k+l+1$ then it is clear that $\beta_n<\delta(X)<\infty$ where $\delta(X)$ is the diameter of X. From (2.3) and by the monotonicity of Ψ we have

(2.4) $$\begin{aligned}\beta_n&\leq\sup_{m\geq 0}\Psi_1[d(x_{n-1},\ x_{n+m-1})]+\sup_{m\geq 0}\Psi_2[d(x_{n-1},\ x_{n+m-1})]\\&\qquad+\sup_{m\geq 0}\Psi_3[d(x_{n-1},\ x_{n+m-1})]\\&\leq\Psi_1[\sup_{m\geq 0}d(x_{n-1},\ x_{n+m-1})]+\Psi_2[\sup_{m\geq 0}d(x_{n-1},\ x_{n+m-1})]\\&\qquad+\Psi_3[\sup_{m\geq 0}d(x_{n-1},\ x_{n+m-1})]\\&\leq\Psi_1(\beta_{n-1})+\Psi_2(\beta_{n-1})+\Psi_3(\beta_{n-1}).\end{aligned}$$

If we take $\beta_{n-1}=0$ for some $n\geq k+l+1$, then we have $x_n=x_{n+1}=fx_n$ i. e., x_n is a fixed point of f. Let $\beta_{n-1}\neq 0$ for all $n\geq k+l+1$. Then from (2.4) and (1.1) we get

$$\beta_n<\beta_{n-1}\ \textit{for}\ n\geq k+l+1,$$

and so the limit $0\leq\beta_\infty=\lim_{n\to\infty}\beta_n<\delta(X)$ exists. Because Ψ is continuous on the right we have $\lim_{n\to\infty}\Psi(\beta_n)=\Psi(\beta_\infty)$ and from (2.4) letting $n\to\infty$

$$\beta_\infty\leq\Psi_1(\beta_\infty)+\Psi_2(\beta_\infty)+\Psi_3(\beta_\infty)<\beta_\infty,$$

if $\beta_\infty>0$, then we have a contradiction so $\beta_\infty=0$ and hence $I(f,\ x_0)$ is Cauchy. Since X is a closed subset of complete metric space we have

(2.5) $$\lim_{n\to\infty}x_n=z\in X.$$

We now that z is a fixed point of f. Let $z\neq fz$. Consider a ball S defined by

(2.6) $$S=\left\{x|x\in X\ ;\ d(x,\ z)\leq\frac{1}{3}d(z,\ fz)\right\}$$

and we see that

(2.7) $$d(x,\ fz)\geq\frac{2}{3}d(z,\ fz),\ x\in S.$$

We choose the smallest integer $N>l+1$ for which $f^N x_0 \in S$. Putting

$$u_1=f^N x_0,\ u_2=z,\ u_3=f^{N+l}x_0,\ u_4=f^{N+k}x_0$$

and considering inequalities (1.2), (2.6) and (2.7) we get

$$\begin{aligned} d[f^{N+1}x_0, fz] &\leq \Psi_1[d(f^N x_0, z)]+\Psi_2[d(f^N x_0, f^{N+k+l}x_0)]+\Psi_3[d(z, f^{N+k+l}x_0)] \\ &\leq \Psi_1[d(f^N x_0, z)]+\Psi_2[d(f^N x_0, z)+d(z, f^{N+k+l}x_0)]+\Psi_3[d(z, f^{N+k+l}x_0)] \\ &\leq \Psi_1\Big[\frac{1}{3}d(z, fz)\Big]+\Psi_2\Big[\frac{1}{3}d(z, fz)+\frac{1}{3}d(z, fz)\Big]+\Psi_3\Big[\frac{1}{3}\ d(z, fz)\Big] \\ &\leq \Psi_1\Big[\frac{1}{2}d(f^{N+1}x_0, fz)\Big]+\Psi_2[d(f^{N+1}x_0, fz)]+\Psi_3\Big[\frac{1}{2}d(f^{N+1}\ x_0, fz)\Big]. \end{aligned}$$

If $f^{N+1}x_0 \neq fz$, then we have a contradiction. Let $f^{N+1}x_0=fz$. Now taking

$$u_1=f^m x_0 (\text{m is a positive integer}),\ u_2=z,\ u_3=f^{N+1-k}x_0,\ u_4=f^{N+1-l}x_0$$

we have from (1.2)

$$\begin{aligned} d(f^{m+1}x_0, fz) &\leq \Psi_1[d(f^m x_0, z)]+\Psi_2[d(f^m x_0, f^{N+1}x_0)]+\Psi_3[d(z, f^{N+1}x_0)] \\ &\leq \Psi_1[d(f^m x_0, z)]+\Psi_2[d(f^m x_0, fz)]+\Psi_3[d(z, fz)], \end{aligned}$$

or $d(z, fz) \leq \Psi_2[d(z, fz)]+\Psi_3[d(z, fz)]$, by letting $m\to\infty$ and by (2.5) which is a contradiction and hence $z=fz$. Next we show the unicity of the fixed point. Let z and w be fixed points of f and $z \neq w$. Then putting $u_1=u_4=z$, $u_3=u_2=w$ in the inequality (1.2) we get

$$0 \leq d(z, w)=d(fz, fw) \leq \Psi_1[d(z, w)]+\Psi_2[d(z, w)]+\Psi_3[d(z, w)]<d(z, w),$$

which is a contradiction. So $z=w$. This completes the proof of the theorem.

3. Remark

We shall show that our theorem contains some well-known results as special cases.

(a) If in the inequality (1.2), we put $\Psi_1=0$ and $\Psi_2=\Psi_3=\phi$ then we get the results of Pittnauer [5].

(b) For $k=l=0$ and $u_3=u_2$ and $u_4=u_1$, $\Psi_i(t)=\phi(t)$ we have Browder [2]. If we define the function $\Psi_i(t)$ by $\Psi_1(t)=az$, $\Psi_2(t)=bz$ and $\Psi_3(t)=cz$, $0\leq z<\infty$ with $a+b+c<1$ then we can prove the theorem assuming X is complete only and this theorem gives the following results as special cases.

(c) For $k=l=1$, $u_3=u_1$, $u_4=u_2$ we get the result of Reich [7].

(d) For $k=l=1$, $u_3=u_1$, $u_4=u_2$ and $a=0$, $b=c=\alpha$ we have Kannan [3].

(e) For $k=l=1$, $u_3=u_2$, $u_4=u_1$ and $a=0$, $b=c=\alpha$ we have Zamfirescu [8].

(f) For $k=l=0$, $u_3=u_1$, $u_4=u_2$ and $b=c=0$, $a=\alpha$ we have Banach [1].

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