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# BAILEY'S FORMULA FOR DOUBLE SERIES

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**1.** Professor Bailey [4. p.245 (Ⅱ.20)] has proved the formula

$$ {}_4F_3\left[\begin{matrix}-n,\ \beta+n,\ \frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha : 1\\ \frac{1}{2}\beta,\ \frac{1}{2}+\frac{1}{2}\beta, 1+\alpha ;\end{matrix}\right]=\frac{(\beta-\alpha)_n}{(\beta)_n} \qquad (1)$$

The author [2] has extended (1) for double series. The object of this paper is to prove another interesting extension of (1) for double series. The result is believed to be new.

The following notation due to Burchnall and Chanundy [3] will be used to represent the hypergeometric series of higher order and of two variables.

$$F\left[\begin{matrix}(a_p) : (b_q) : (c_r) : x, y\\ (d_s) : (e_k) : (f_h) :\end{matrix}\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a_p)]_{m+n}[(b_q)]_m[(c_r)]_n x^m y^n}{[(d_s)]_{m+n}[(e_k)]_m[(c_r)]_n m!n!}, \qquad (2)$$

where $(a_p)$ and $[(a_p)]_{m+n}$ will mean the sequence $a_1,\cdots,a_p$ and the product $(a_1)_{m+n}\cdots(a_p)_{m+n}$.

In the investigation we require the result due to Saalschutz [4, p.243 (Ⅱ.2)]

$$ {}_3F_2\left[\begin{matrix}-n,\ \alpha,\ \beta : 1\\ \gamma,\ 1+\alpha+\beta-\gamma-n ;\end{matrix}\right]=\frac{(\gamma-\alpha)_n(\gamma-\beta)_n}{(\gamma)_n(\gamma-\alpha-\beta)_n}, \qquad (3)$$

and Dougall [4, p.244 (Ⅲ.13)]

$$ {}_5F_4\left[\begin{matrix}\alpha,\ 1+\frac{1}{2}\alpha,\ \beta,\ -m,\ -n : 1\\ \frac{1}{2}\alpha,\ 1+\alpha-\beta,\ 1+\alpha+m,\ 1+\alpha+n :\end{matrix}\right]$$

$$=\frac{(1+\alpha)_n(1+\alpha)_m(1+\alpha-\beta)_{m+n}}{(1+\alpha-\beta)_n(1+\alpha-\beta)_m(1+\alpha)_{m+n}}. \qquad (4)$$

**2.** The formula to be proved is

$$F\left[\begin{matrix}\frac{1}{2}\alpha : -n,\ \alpha+\beta_1+n,\ \frac{1}{2}+\frac{1}{2}\alpha : -m,\ \alpha+\beta_2+m,\ \frac{1}{2}+\frac{1}{2}\alpha : 1;\ 1\\ 1+\alpha : \frac{1}{2}(\alpha+\beta_2),\ \frac{1}{2}(1+\alpha+\beta_2) : \frac{1}{2}(\alpha+\beta_2), \frac{1}{2}(1+\alpha+\beta_2) :\end{matrix}\right]$$

$$=\frac{(\beta_2)_{n-m}}{(\beta_2+\alpha)_{n-m}}=\frac{(\beta_2)_{m-n}}{(\beta_2+\alpha)_{m-n}}, \tag{5}$$

provided that $\alpha+\beta_1+\beta_2=1$.

PROOF. To prove (5), we start with the left side of (5)

$$F\left[\begin{matrix}\frac{1}{2}\alpha : -n,\ \alpha+\beta_1+n,\ \frac{1}{2}+\frac{1}{2}\alpha : -m,\ \alpha+\beta_2+m,\ \frac{1}{2}+\frac{1}{2}\alpha : 1,\ 1\\ 1+\alpha : \frac{1}{2}(\alpha+\beta_1),\ \frac{1}{2}(1+\alpha+\beta_1) : \frac{1}{2}(\alpha+\beta_2),\ \frac{1}{2}(1+\alpha+\beta_2) :\end{matrix}\right]$$

$$=\sum_{p=0}^{n}\sum_{q=0}^{m}\frac{\left(\frac{1}{2}\alpha\right)_{p+q}(-n)_p(\alpha+\beta_1+n)_p\left(\frac{1}{2}+\frac{1}{2}\alpha\right)_p(-m)_q(\alpha+\beta_2+m)_q\left(\frac{1}{2}+\frac{1}{2}\alpha\right)_q}{(1+\alpha)_{p+q}\left(\frac{1}{2}\alpha+\frac{1}{2}\beta_1\right)_p\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta_1\right)_p\left(\frac{1}{2}\alpha+\frac{1}{2}\beta_2\right)_q}$$
$$\times\frac{1}{\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta_2\right)_q p!q!}$$

by (2)

$$=\sum_{p=0}^{n}\sum_{q=0}^{m}\frac{(-n)_p(\alpha+\beta_1+n)_p\left(\frac{1}{2}\alpha\right)_p\left(\frac{1}{2}+\frac{1}{2}\alpha\right)_p}{\left(\frac{1}{2}\alpha+\frac{1}{2}\beta_1\right)_p\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta_1\right)_p(1+\alpha)_p p!}$$
$$\frac{(-m)_q(\alpha+\beta_2+m)_q\left(\frac{1}{2}\alpha\right)_q\left(\frac{1}{2}+\frac{1}{2}\alpha\right)_q}{\left(\frac{1}{2}\alpha+\frac{1}{2}\beta_2\right)_q\left(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\beta_2\right)_q(1+\alpha)_q q!}$$
$$\sum_{r=0}^{\min(p,q)}\frac{(\alpha)_r\left(1+\frac{1}{2}\alpha\right)_r\left(1+\frac{1}{2}\alpha\right)_r(-p)_r(-q)_r}{\left(\frac{1}{2}\alpha\right)_r\left(\frac{1}{2}\alpha\right)_r(1+\alpha+p)_r(1+\alpha+q)_r r!} \quad \text{by (4).}$$

$$=\sum_{r=0}^{\min(p,q)}\frac{(-n)_r(-m)_r(\alpha+\beta_1+n)_r(\alpha+\beta_2+m)_r(\alpha)_r}{(\alpha+\beta_1)_{2r}(\alpha+\beta_2)_{2r}r!}$$
$${}_4F_3\left[\begin{matrix}-n+r,\ \alpha+\beta_1+n+r,\ \frac{1}{2}\alpha+r,\ \frac{1}{2}+\frac{1}{2}\alpha+r : 1\\ \frac{1}{2}(\alpha+\beta_1+2r),\ \frac{1}{2}(\alpha+\beta_1+1+2r),\ 1+\alpha+2r :\end{matrix}\right]$$
$${}_4F_3\left[\begin{matrix}-m+r,\ \alpha+\beta_2+m+r,\ \frac{1}{2}\alpha+r,\ \frac{1}{2}+\frac{1}{2}\alpha+r : 1\\ \frac{1}{2}(\alpha+\beta_2+2r),\ \frac{1}{2}(\alpha+\beta_2+1+2r),\ 1+\alpha+2r :\end{matrix}\right]$$

$$=\frac{(\beta_1)_n(\beta_2)_m}{(\beta_1+\alpha)_n(\beta_2+\alpha)_m}\,{}_3F_2\left[\begin{matrix}-m,\ -n,\ \alpha : 1\\ 1-\beta_2-m,\ 1-\beta_1-n :\end{matrix}\right] \quad \text{by (1)}$$

$$=\frac{(\beta_1)_{n-m}}{(\beta_1+\alpha)_{n-m}}=\frac{(\beta_2)_{m-n}}{(\beta_2+\alpha)_{m-n}}, \qquad \text{by (3)}$$

provided that $\alpha+\beta_1+\beta_2=1$. This completes the proof of (5). In case $n=0$ or $m=0$ in (5), it reduces to (1), Thus we call (5) the extension of Bailey's formula for double series.

**3.** Now we multiply both sides of (5) by $(-x)^n(-y)^m(\alpha+\beta_1)_n(\alpha+\beta_2)_m\frac{1}{n!}\frac{1}{m!}$ and summing from $n=0$ to $\infty$ and $m=0$ to $\infty$, after a little calculation, we get

$$G_2(\alpha+\beta_1,\ \alpha+\beta_2,\ \beta_2,\ \beta_1 : x,\ y)=(1+x)^{-\alpha-\beta_1}(1+y)^{-\alpha-\beta_2}$$
$$F_1\left[\frac{1}{2}\alpha : \frac{1}{2}+\frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha : 1+\alpha : \frac{4x}{(1+x)^2},\ \frac{4y}{(1+y)^2}\right], \tag{6}$$

where $\alpha+\beta_1+\beta_2=1$ and $G_2$ is a Hern function [1] defined as follows

$$G_2(\alpha,\ \alpha',\ \beta,\ \beta' : x,\ y)$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(\alpha)_m(\alpha')_n(\beta)_{n-m}(\beta')_{m-n}\frac{x^m}{m!}\ \frac{y^n}{n!}. \tag{7}$$

Now we use the formula due to Erdelyi [1, p. 149, (21)]

$$G_2(\alpha,\ \alpha',\ \beta,\ \beta' ; x,\ y)=(1+x)^{-\alpha}(1+y)^{-\alpha'}$$
$$F_2\left[1-\beta-\beta' ; \alpha,\ \alpha' ; 1-\beta,\ 1-\beta' ; \frac{x}{1+x},\ \frac{y}{1+y}\right] \tag{8}$$

in (6), we get a new formula

$$F_1\left[\frac{1}{2}\alpha : \frac{1}{2}+\frac{1}{2}\alpha,\ \frac{1}{2}+\frac{1}{2}\alpha : 1+\alpha : \frac{4x}{(1+x)^2},\ \frac{4y}{(1+y)^2}\right]$$
$$=\left[1-\frac{x}{1+x}-\frac{y}{1+y}\right]^{-\alpha}.$$

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