Kyungpook Math. J.
Volume 19, Number 2
December, 1979

ON FOULIS PAPER

By S.H. Raza

This paper is based on the results of D.J. Foulis paper '*Relative Inverses in Baer-∗-Semigroups*' [1]. We shall follow the notation and terminology given in this paper. For the sake of completeness we are giving the following definitions. A ∗-semigroup is a semigroup S with an involutorial antiautomorphism $x \longrightarrow x^*$ such that

(i) $(xy)^*=y^*x^*$ and (ii) $x^{**}=x$ for all x, y in S.

A projection in such an S is an element e in S with $e=e^2=e^*$. The partially ordered set of all projections in S is denoted by $P(S)$, the partial order being defined by $e \leqq f$ if and only if $e=ef$ $(e,f \in P(S))$

A *Baer-∗-semigroup* is ∗-semigroup S with a two sided zero 0 with the property: for each element a in S there exists a projection $a' \in P(S)$ such that $\{x \in S \mid ax=0\}=a's$. We define $P'(S)=P(S)$ by the condition $P'(S)=\{a' \mid a \in S\}$. A projection e in $P(S)$ is said to be *closed* if $e=e''$. By (v) of theorem 1 [1] a projection e is closed if and only if $e=a'$ for some a in S. An element a in S is said to be *right ∗-regular* in S if $aS=(a^*)''S$, a is *left ∗-regular* in S if $Sa=Sa''$. If a in S is both right and left ∗-regular in S, then a is said to be *∗-regular* in S.

A slight different, but equivalent definition of ∗-regular in S as defined that if a is an element of the Baer-∗-semigroup S, then a is *∗-regular* in S if there exists a unique element a^+ in S such that $a=aa^+a$, $a^+=a^+aa^+$, $aa^+=(a^*)''$ and $a^+a=a''$. An element a in S is *range closed* if the condition g in $P'(S)$ with $g \leqq a''$ and $(ga^*)''=(a^*)''$ necessarily implies $g=a''$. a^+ is relative inverse of a.

In an involution semigroup S, let $e=e^2=e^* \in S$ and $f=f^2=f^* \in S$. If there exists an element $x \in S$ such that $x=exf$, $x^*x=f$, $xx^*=e$, then we say that e and f are *∗-equivalent* and we write $x: e \sim^* f$ and x is partially unitary element of S.

In this note we give some interesting results which are consequences of the beautiful results given in [1].

THEOREM 1. *If a is ∗-regular element in S, then*

$$[g'\wedge((a^+)^*)''a^+]''=(ga^*)'\wedge(a^*)''.$$

PROOF. Let $h=(ga^*)'$. Then $h'=(ga^*)''\leqq(a^*)''$. (By thm. 1 (xiv) [1]) and $hC(a^*)''$. Hence $ha=(h\wedge(a^*)'')a$ by lemma 5 [1].

$$(ha)''=[(h\wedge(a^*)'')a]''$$

$$(h(a^*)''a)''=[(h\wedge(a^*)'')a]''.$$

$$[(h(a^*)'')a]''=[[(h\wedge(a^*)'')a]''a] \text{ by (xii) of thm. 1 [1].}$$

$$(h(a^*)'')''=h\wedge(a^*)'' \text{ by (v) of thm 6 [1].}$$

L. H. S. $((ga^*)'(a^*)'')''=((ga^*)'aa^+)''=[[((ga^*)'a)''a^+]''a^+]$ by (xii) of thm. 1 [1]

$=[(g'\wedge a'')a^+]''$ by thm. 6 [1]

$=[(g'\wedge((a^+)^*)''a^+]''$ by cor. of thm. 11 [1].

Therefore $[(g'\wedge((a^+)^*)''a^+]''=(ga^*)'\wedge(a^*)''$.

THEOREM 2. *If a is ∗-regular in a Baer-∗-semigroup S,* $g\leqq a''$, $(ga^*)''=(a^*)''$ *and* $a''\nless g\lneqq 1$, $1\neq g\in P'(S)$, *then* $(ga^+)''=(a^*)''$.

PROOF. a is ∗-regular in S implies a is range closed in S, by lemma. 8 [1]. Then there exists g in $P'(S)$ such that $g=a''$. Now $ga''=a''a''=a''$. So $ga''=a''$ which gives by cor. of thm 11 [1]

$$((ga^+)''a)''=((a^+)''a)''$$

Further by thm. 6 and cor. of thm. 11.

$$(ga^+)''=(a^+)''=(a^*)''$$

since $(ga^+)''\leqq(a^+)''$ by (xiv) of thm. 1 [1] and $(a^+)''=(a^*)''$.

THEOREM 3. *Let a be ∗-regular in a Baer-∗-semigroup S and let* a^+ *be its relative inverse. Then*

$$(e'a^*)'\wedge(a^*)''=(ea^+)'' \text{ for } 1\neq e\in P'(S).$$

PROOF. Let $f=((e'a^*)''\vee(a^*)')'$ …… (i). Then $fa=((e'a^*)'\wedge(a^*)'')a$ and $(fa)''=(((e'a^*)'\wedge(a^*)'')a)''$. Since $(e'a^*)''\leqq(a^*)''$ by (xiv) of thm. 1 [1], so $(e'a^*)'C(((a^*)'')a)$ by lemma. 5 [1], we get $(fa)''=((e'a^*)'a)''$ which gives $(fa)''=e\wedge a''=ea''$ by thm. 6 [1] and 36.6 [1], since a is range closed by lemma 8 [1]. Hence $(fa)''=ea''=(ea^+a)''=((ea^+)''a)''$ by cor. of thm. 11 [1] and (xii) of thm. [1]. By (i) $f\leqq(a^*)''$, $(ea^+)''\leqq(a^+)''=(a^*)''$ by (v) of cor. of thm. 11 [1]. We have $f=(ea^+)''$ by thm. 6 (v) [1]. Therefore $(e'a^*)'\wedge(a^*)''=(ea^+)''$.

THEOREM 4. *Let a be ∗-regular and* a^+ *be its inverse in a Baer-∗-semigroup.*

If $a: g\sim^{*} e$ *and* $(e\wedge g)\, a''=ea''$ *then* $[((e\wedge g)a^{+})''\wedge g'] = a'(ea^{+})$, $1\neq g\in P'(S)$, $1\neq e\in P'(S)$.

PROOF. As $(e\wedge g)a''=ea''$ which gives by cor. of thm. 11 [1]

$$((e\wedge g)a^{+}a)''=(ea^{+}a)''.$$ Now by (xii) of thm. 1 [1]

we have $(((e\wedge g)a^{+})''a)''=((ea^{+})''a)''$. By thm. 6 and cor. of thm. 11 [1] gives $((e\wedge g)a^{+})''=((ea^{+})'')$.

Since $((e\wedge g)a^{+})''\leqq(a^{+})''=(a^{*})''$ and $(ea^{+})''\leqq(a^{+})''=(a^{*})''$

$$((e\wedge g)a^{+})''\wedge g'=((ea^{+})''\wedge g')$$

$$((e\wedge g)a^{+})''\wedge g'=[(g(a^{+})^{*}e)'(ea^{+})]$$ by thm. 6 (i) [1].

Since a is range closed by lemma 8.

Now $aa^{+}=(a^{*})''$ by cor. of thm. 11 [1], so $aa^{+}g=(a^{*})''g$ which implies

$$(a^{+}g)^{*}a^{*}=g(a^{*})''.$$

We have $(g(a^{+})^{*}a^{*})=g(a^{*})''$,

$$(g(a^{+})^{*})a^{*}a=g(a^{*})''\, a.$$

Therefore $(g(a^{+})^{*})e=g(a^{*})''\, a$, since $a^{*}a=e$

by def. of $*$-equivalent.

We get $((e\wedge g)a^{+})''\wedge g'=[(g(a^{*})''a)'(ea^{+})]$

$=[gaa^{+}a]'(ea^{+})$ by (ii) of cor. of thm. 11 [1]

$=(ga)'(ea^{+})$.

Since $ga=a$ (because $aa^{*}=(ga)a^{*}=(ga)(ga)^{*}$ by $*$-cancellation law $ga=a$), hence

$$((e\wedge g)a^{+})''\wedge g'=a'(ea^{+})$$

THEOREM 5. *Let* a *be* $*$*-regular in a Baer-*$*$*-semigroup* S, a^{+} *be its relative inverse,* $h\leqq a''$ *and* $(e\vee g)\leqq a''$, *and* $(e,f)M$ $\forall e,f\in P'(S)$. *Then* $h=(e'g')'$ *if and only if*

$$((ha^{*})''\wedge(a^{*})'')a=((e\wedge g)'a^{*})''a,\ h,\ g\in P'(S).$$

PROOF. If $h=(e'g')'$. Then $h=(e'\wedge g')'$. So $(ha^{*})''a=((e'\wedge g')a^{*})''\, a$.

By(xiv) of thm. 1 [1], $h(a^{*})''\leqq(a^{*})''$. So $h(a^{*})''C(a^{*})''$.

By lemma 5 $((ha^{*})''a)=((ha^{*})''\wedge(a^{*})'')a$. We have

$$((ha^{*})''\wedge(a^{*})'')a=((e\wedge g)'a^{*})''a.$$

Conversely if $((ha^{*})''\wedge(a^{*})'')a=((e'\wedge g')a^{*})''a$. Since $(ha^{*})''\wedge(a^{+})''\leqq(a^{+})''=(a^{*})''$ and $((e'\wedge g')'a^{*})''\leqq(a^{*})''$ by (xiv) of thm. 1 [1], hence $(ha^{*})''\wedge(a^{+})''=((e'\wedge g')'a^{*})''$ by thm. 6 as a is range closed by lemma 8.

$(ha^*)''(a^+)''=(ha^*)''(a^*)''=((e'\wedge g')'a^*)''$ by 37.7 [2] and (v) cor. of thm. 11 [1]. As $(ha^*)''\leqq(a^*)''$, so $(ha^*)''(a^*)''=(ha^*)''$, $(ha^*)''=((e'\wedge g')'a^*)''$
By thm. 6, $h\vee a'=(e'\wedge g')'\vee a'$ because a is range closed by lemma 8. We have

$$(h\vee a')\wedge a''=((ee'\wedge g')'\vee a')\wedge a''.$$

So $h=(e'\wedge g')'=(e'g')'$ by thm. 37.7 [2] since $(e, f)M$ $\forall e, f\in M$.

THEOREM 6. *If a is $*$-regular in a Baer-$*$-semigroup and a^+ is relative inverse, then*

$$[(\{(g'\wedge a'')a^*\}'\wedge(a^*)'')a]''=[((ga^+)''\wedge(a^*)'')a]''\wedge(a^*)''.$$

PROOF. Since $(ga^+)''\leqq(a^+)''=(a^*)''$ by (xiv) of thm. 1 and (v) of cor. of thm. 11 [1], then $(ga^+)''\,C(a^*)''$. So by lemma 5,

$$((ga^+)''a)''=(((ga^+)''\wedge(a^*)'')a)''$$
$$(ga^+a)''=(((ga^+)''\wedge(a^*)'')a)'' \text{ by (xii) of thm. 1}$$
$$(ga'')''=(((ga^+)''\wedge(a^*)'')a)'' \text{ by (i) of cor. of thm. 11.}$$

By (xv) of Thm 1 [1], $(ga'')''=(g\vee a')\wedge a''$.

$$=[\{(g'\wedge a'')a^*\}'a] \text{ by thm. 6 [1],}$$

since a is range closed by lemma 8 [1].
Let $h=\{(g'\wedge a'')a^*\}'$. Then $h'\leqq(a^*)''$, hence $h'C(a^*)''$ and $h'a=[(h'\wedge(a^*)'')\ a]$ by lemma 5 [1]
Now $(h'a)''=\{(h'\wedge(a^*)'')a\}''$.
So on putting the value of h.

$$[\{(((g'\wedge a'')a^*)'a^*)'\wedge(a^*)''\}a]''=[((ga^+)''\wedge(a^*)'')a]''\wedge(a^*)''$$

THEOREM 7. *Let a is $*$-regular in a Baer-$*$-semigroup, a^+ is its relative inverse in S, $g\leqq a''$, $(ga^*)''=(a^*)''$, $a''\not< g\leqq 1$ and $eC(a^*)''$. Then*

(*i*) $([(e\wedge(ga^*)'')a]^*)''=[(a^+e)]''$

(*ii*) $[e(a^*)'']''=[(g\vee a')\wedge(a^*)''\wedge e]$.

PROOF. As $(ga^*)''=(a^*)''$, so $([(e\wedge(a^*)'')a]^*)''=([(e\wedge(ga^*)'')a]^*)''$
Hence by lemma 5 [1] we get
$([(e\wedge(ga^*)'')a]^*)''=((ea)^*)''=(a^*e)''=((a^*)''e)''$ by (xii) of thm. 1 [1] which implies
$([(e\wedge(ga^*)''a]^*)''=((a^+)''e)''=(a^+e)''$ by (xii) of thm. 1 [1].

Proof of (ii). Since a is range closed in S by lemma 8 [1]. Hence there exists g in $P'(S)$ such that $g=a''$. Now $0=g'\wedge g=g'\wedge a''$ which gives $1=g\vee a'$. $1\wedge$

$(a^*)''=(g\vee a')\wedge(a^*)''$ and so $a=[e\wedge(a^*)'']a=([(g\vee a')\wedge(a^*)''\wedge e]a)$ by lemma 5 [1]. Further $[e(a^*)''a]''=(ea)''=(\{(g\vee a')\wedge(a^*)''\wedge e\}a)''$, since $a=aa^+a=(a^*)''a$. Finally we get $[e(a^*)'']''=[(g\vee a')\wedge(a^*)''\wedge e]$, since $(g\vee a')\wedge(a^*)''\wedge e\leqq(a^*)''$, $[e(a^*)''a]''=[e(a^*)''a]''$, and $(e(a^*)'')''\leqq(a^*)''$ by (xii) and (xiv) of thm. 1 [1].

The author is grateful to Professor D. J. Foulis for his valuable comments and suggestions.

C/O Dr. V. N. Dixit
Department of Mathematics
University of Delhi Delhi-110007
India

REFERENCES

[1] Foulis, D. J., *Relative inverses in Baer-*-semigroups*, Michigan Mathematical Journal, Vol. 10, (1963), pp. 65—84.

[2] Maeda, F. and Maeda, S., *Theory of symmetric lattices*, Springer Verlag, Berlin, Heidelberg, New York, 1970.