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SOLVABLE OPERATOR SEMIGROUPS WITH AN APPLICATION TO SHORT EXACT SEQUENCES

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1. Introduction

When an attempt is made to generalize certain theorems concerning groups to semigroups, it is reasonable to expect that additional conditions are necessary. P.J. Allen and W.R. Windham [1] proved the isomorphism theorems for certain operator semigroups and used those results to prove Schreier's refinement theorem and the Jordan-Holder theorem. In order to obtain those results, they used the concept of relatedness among subsemigroups and considered homomorphisms that were semimaximal. The purpose of this paper is to extend their results to solvable operator semigroups and make an application to short exact sequences. An example is given to illustrate these concepts.

2. Basic definitions and fundamentals

The definitions used in this paper are basically the same as those appearing in [1]. Throughout this paper, quotients will be structured with respect to the congruence defined by Dubreil [2].

DEFINITION 2.1. An operator semigroup is a triple $(G, M, \cdot)$ consisting of an associative semigroup G with an identity 1, an operator set M, and a mapping of $G\times M$ into G such that

(1) $(g_1g_2)m=(g_1m)(g_2m)$, (2) $1m=1$

for each $g_i\in G$ and $m\in M$. This system will be called an *M-semigroup G.*

DEFINITION 2.2. A subset H of an M-semigroup G is called an *M-subsemigroup of G* if H is itself an M-semigroup with respect to M and the operation in G.

DEFINITION 2.3. An M-subsemigroup H of an M-semigroup G is called a *k-subsemigroup* of G if $g\in G$, $h\in H$ and $gh\in H$ imply that $g\in H$.

If the operation in G is addition, then H is a k-M-subsemigroup of G if $g\in G$, $h\in H$ and $g+h\in H$ imply that $g\in H$. The k-M-subsemigroups were very useful in the development of the isomorphism theorems for operator semigroups.

DEFINITION 2.4. An M-subsemigroup H of an M-semigroup G is said to be *normal* in G, denoted $H \lhd G$, if $Hg = gH$ for all $g \in G$.

The notions of M-homomorphism, kernel, and isomorphism will be the same as those used for operator groups.

DEFINITION 2.5. A homormophism η from an M-semigroup G_1 to an M-semigroup G_2 will be called *semimaximal* if $g_1\eta = g_2\eta$ implies that $g_1 \ker \eta \cap g_2 \ker \eta \neq \phi$.

DEFINITION 2.6. Let H_1 and H_2 be M-subsemigroups of an M-semigroup G. H_1 is related to H_2 provided $g_i, g_i' \in H_i$ and $g_1 g_2 = g_1' g_2'$ imply the existence of elements $a, b \in H_1 \cap H_2$ such that $g_1 a = g_1' b$. H_1 is said to be *closely related to* H_2 if H_1 is related to every M-subsemigroup of H_2.

It is clear that an M-semigroup is closely related to each of its M-subsemigroups and if H_1 and H_2 are M-subsemigroups of an M-semigroup G such that $H_2 \subset H_1$, then H_1 and H_2 are closely related.

The concepts of relatedness among M-subsemigroups of an M-semigroup and semimaximal homomorphism were the additional conditions necessary to enable Allen and Windham to extend the isomorphism theorems to operator semigroups.

The following theorems may be found in [1]. They are crucial to the study of solvable M-semigroups. Consequently, they are stated here for completeness.

THEOREM 2.7. (Fundamental theorem of homomorphisms). *Let η be a semimaximal homomorphism of an M-semigroup G_1 onto an M-semigroup G_2 such that* $\ker \eta \lhd G_1$. *Then* $G_1/\ker \eta \cong G_2$.

THEOREM 2.8. (Correspondence theorem). *Let η be a semimaximal homomorphism from an M-semigroup G_1 onto an M-semigroup G_2 and $\{H_\alpha\}$ the collection of all k-M-subsemigroups of G_1 that contain* $\ker \eta$. *The mapping $H \to H\eta$ is one-to-one of $\{H_\alpha\}$ onto the collection of all k-M-subsemigroups of G_2.*

THEOREM 2.9. (Lattice theorem). *Let G be an M-semigroup and H a normal k-M-subsemigroup of G. Then any k-M-subsemigroup of G/H is of the form N/H, where N is a k-M-subsemigroup of G containing H. If N_1 and N_2 are distinct k-M-subsemigroups of G containing H, then N_1/H and N_2/H are distinct k-M-subsemigroups of G/H. If $N \lhd G$, then $N/H \lhd G/H$.*

THEOREM 2.10. (First isomorphism theorem). *Let η be a semimaximal homomorphism from an M-semigroup G_1 onto an M-semigroup G_2. Let H be a normal k-M-subsemigroup of G_1 that contains* ker η. *Then $G_1/H \cong G_2/H\eta$.*

COROLLARY 2.11. *If N and H are normal k-M-subsemigroups of an M-semigroup G such that $N \subset H$, then*

$$\frac{G}{H} \cong \frac{G/N}{H/N}.$$

THEOREM 2.12. (Second isomorphism theorem). *Let G_1 and G_2 be M-subsemigroups of an M-semigroup G such that G_1 is related to G_2, $G_2 \lhd G_1G_2$, and G_1, G_2 are k in G_1G_2. Then*

(1) $(G_1 \cap G_2) \lhd G_1$, (2) $G_1G_2/G_2 \cong G_1/G_1 \cap G_2$.

3. Solvable M-semigroups

Solvable groups are usually applied to Galois theory. However, they form a large class of groups of purely group-theoretical interest as well. Consequently, the solvable semigroup is also interesting. The concept of a solvable M-semigroup will be the same as that of a solvable group. However, to establish the results about solvable M-semigroups that are valid for solvable groups, attention must be paid to the conditions placed on the normal M-subsemigroups.

DEFINITION 3.1. A normal series for an M-semigroup G is a chain of M-subsemigroups

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = \{1\}$$

such that $G_{i+1} \lhd G_i$ for each i. The quotients G_i/G_{i+1} are called *the factors of the series* and the number of strict inclusions is called the *length of the series*. If each G_{i+1} is a k-subsemigroup of G_i, the series is called *a normal k-series.*

DEFINITION 3.2. A normal series

$$G = G_0 \supset G_1 \supset \cdots \supset G_n = \{1\}$$

is called *solvable* if it is a k-series with abelian factors.

DEFINITION 3.3. An M-semigroup G is said to be *solvable* if it has a solvable series.

Now we are ready to extend some of the results concerning solvable groups to solvable semigroups.

THEOREM 3.4. *If G is a solvable M-semigroup and H is an M-subsemigroup* *of G, then H is solvable.*

PROOF. Let $G=G_0\supset G_1\supset\cdots\supset G_n=\{1\}$ be a solvable series for G and consider the series

$$H=H\cap G_0\supset H\cap G_1\supset\cdots\supset H\cap G_n=\{1\}.$$

Since $G_{i+1}\subset G_i$, it is clear that G_i is closely related to G_{i+1} for each i. Thus, $H\cap G_i$, being an M-subsemigroup of G_i, is related to G_{i+1}. Now it follows from the series being a normal k-series, that $G_{i+1}\lhd(H\cap G_i)G_{i+1}$ and that $H\cap G_i$, G_{i+1} are k in $(H\cap G_i)G_{i+1}$.

Consequently, by theorem 2.12. (Second isomorphism theorem), $H\cap G_{i+1}=(H\cap G_i)\cap G_{i+1}\lhd H\cap G_i$ and

$$\frac{H\cap G_i}{H\cap G_{i+1}}=\frac{H\cap G_i}{(H\cap G_i)\cap G_{i+1}}\cong\frac{(H\cap G_i)G_{i+1}}{G_{i+1}}\subset\frac{G_i}{G_{i+1}}.$$

Since G_i/G_{i+1} is abelian, it follows that $H\cap G_i/H\cap G_{i+1}$ is abelian. Consequently, the series for H is a normal k-series with abelian factors. Therefore H is a solvable M-semigroup.

The above theorem indicates that solvable M-semigroups may be formed from M-subgroups of known solvable M-semigroups. The next theorem tells us when the homomorphic image of a solvable M-semigroup is a solvable M-semigroup.

THEOREM 3.5. *Let G and $\bar{G}$ be M-semigroups and $\theta: G\to G$ be an onto semimaximal homomorphism. If G has a solvable series such that* ker θ *is contained in each non-trivial term of the series, then $\bar{G}$ is solvable.*

PROOF. Let $G=G_0\supset G_1\supset\cdots\supset G_n=\{1\}$ be a solvable series for G such that ker θ is contained in each non-trivial term of this series and consider the series $\bar{G}=\bar{G}_0\supset\bar{G}_1\supset\cdots\supset\bar{G}_n=\{1\}$, where $\bar{G}_i=G_i\theta$. By theorem 2.8 (Correspondence theorem), this series is a normal k-series. By theorem 2.10 (First isomorphism theorem), $G_i/G_{i+1}\cong G_i\theta/G_{i+1}\theta=\bar{G}_i/\bar{G}_{i+1}$. Since G_i/G_{i+1} is abelian, so is $\bar{G}_i/\bar{G}_{i+1}$. Consequently, the series is a solvable series and $\bar{G}$ is solvable.

THEOREM 3.6. *Let G be a solvable M-semigroup and H is a normal M-subsemigroup of G. If G has a solvable series in which H is a term, then G/H is solvable.*

PROOF. Let $\eta: G\to G/H$ be the natural map. Then η is a semimaximal homomorphism. Now let

$$G=G_0\supset G_1\supset\cdots\supset G_n=H\supset G_{n+1}\supset\cdots\supset G_p=\{1\}$$

be the solvable series containing H. Then H is k in G, since this series is solvable, and it follows that ker $\eta=H$. Since η is onto, theorem 2.8 (Correspondence theorem) gives a one-to-one correspondence between the k-M-subsemigroups of G containing H and the k-M-subsemigroups of G/H. Now by theorem 2.9 (Lattice theorem),

$$G/H=G_0/H\supset G_1/H\supset\cdots\supset G_n/H=H/H=\{1\}$$

is a normal k-series for G/H. It follows from Corollary 2.11 that

$$G_i/G_{i+1}\cong\frac{G_i/H}{G_{i+1}/H}.$$

Consequently, $\dfrac{G_i/H}{G_{i+1}/H}$ is abelian and the series is solvable. Therefore G/H is solvable.

THEOREM 3.7. *If G is an M-semigroup and H is a normal M-subsemigroup of G such that H and G/H are solvable, then G is solvable.*

PROOF. Since H and G/H are solvable, there are solvable series

$$G/H\supset\overline{M}_1\supset\overline{M}_2\supset\cdots\supset\overline{M}_n=\{1\}=H$$

and

$$H=H_0\supset H_1\supset\cdots\supset H_m=\{1\}.$$

By the Lattice theorem, each $\overline{M}_i$ is of the form M_i/H where M_i is a normal k-M-subsemigroup of G containing H. Since $\overline{M}_{i+1}\lhd\overline{M}_i$, it follows that $M_{i+1}\lhd M_i$. Now $\overline{M}_i/\overline{M}_{i+1}$ is abelian and it follows from corollary 2.11 that

$$\overline{M}_i/\overline{M}_{i+1}=\frac{M_i/H}{M_{i+1}/H}\cong M_i/M_{i+1}$$

and M_i/M_{i+1} is abelian. Hence

$$G=M_0\supset M_1\supset\cdots\supset M_n=H$$

is a solvable series from G to H. Consequently,

$$G=M_0\supset M_1\supset\cdots\supset M_n=H\supset H_1\supset\cdots\supset H_m=\{1\}$$

is a solvable series for G. Therefore, G is solvable.

THEOREM 3.8. *Let G_1 and G_2 be solvable M-subsemigroups of G, G_1 and G_2 be related, $G_2\lhd G_1G_2$, and G_1, G_2 be k in G_1G_2. If G_1 has a solvable series in which $G_1\cap G_2$ is a term, then G_1G_2 is solvable.*

PROOF. By the second isomorphism theorem, $G_1\cap G_2\lhd G_1$ and $G_1G_2/G_2\cong G_1/G_1\cap G_2$. By theorem 3.6, $G_1/G_1\cap G_2$ is solvable. Thus G_1G_2/G_2 is solvable. But G_2 is solvable and $G_2\lhd G_1G_2$. Consequently, theorem 3.7 gives that G_1G_2 is solvable.

These theorems indicate that under certain conditions we can construct solvable semigroups in practically the same manner as solvable groups are constructed.

4. Short exact sequences of M-semigroups

For the remainder of the paper all semigroups will be written additively and the trivial semigroup having one element is denoted by 0 instead of $\{0\}$.

DEFINITION 4.1. Let

$$\cdots \to G_{k+1} \xrightarrow{\alpha_{k+1}} G_k \xrightarrow{\alpha_k} G_{k-1} \xrightarrow{\alpha_{k-1}} G_{k-2} \to \cdots$$

be a sequence of M-semigroups and semimaximal homomorphisms with normal kernels. The sequence is called *an exact sequence* if Im $\alpha_{i+1}=\ker \alpha_i$ for each i. A short exact sequence is an exact sequence of the form

$$O \to G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \to O.$$

Observe that we are requiring that all homomorphisms be semimaximal.

LEMMA 4.2. *Let*

$$O \to G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \to O$$

be a short exact sequence. Then α is one-to-one and β is onto.

PROOF. Since $O \to G_1 \xrightarrow{\alpha} G_2$ is exact, $\ker \alpha=O$. Now suppose $g_1\alpha=g_2\alpha$. Since α is semimaximal, $g_1+\ker \alpha \cap g_2+\ker \alpha \neq \phi$. But $\ker \alpha=O$. Consequently $g_1+O \cap g_2+O=\{g_1\}\cap\{g_2\}\neq\phi$ and it follows that $g_1=g_2$. Therefore α is one-to-one. Since $G_2 \xrightarrow{\beta} G_3 \to O$ is exact, Im $\beta=\ker O=G_3$ and β is onto.

THEOREM 4.3. *Let*

$$O \to G_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \to O$$

be an exact sequence. Then

(*i*) *If G_2 has a solvable series containing* im α, *then G_1 and G_3 are solvable.*

(*ii*) *If G_1 and G_3 are solvable, then G_2 is solvable.*

PROOF. (i) Suppose G_2 has a solvable series containing Im α. Now by lemma 4.2, α is one-to-one. By exactness, Im $\alpha=\ker \beta$. Consequently, $G_1\cong\ker \beta$. But $\ker \beta \lhd G_2$ and it follows from theorem 3.4 that $\ker \beta$ is solvable. Therefore

G_1 is solvable. By lemma 4.2, β is onto. Thus theorem 2.7 gives that $G_2/\ker\beta \cong G_3$. But theorem 3.6 gives that $G_2/\ker\beta$ is solvable. Therefore G_3 is solvable.

(ii) Suppose G_1 and G_3 are solvable. Again we have $G_1 \cong \ker \beta$ and $G_2/\ker \beta \cong G_3$. Consequently, theorem 3.7 gives that G_2 is solvable.

We now construct an example of a solvable M-semigroup and an example of an exact sequence of M-semigroups.

EXAMPLE 4.4. Let $K=\{x|x \text{ is real and } 0\leq x\leq 1\}$, $H=\{x|x \text{ is real and } 1<x\leq 2\}$, and $G=K\cup H$. Define multiplication in G as follows: If k_1, $k_2\in K$ and h_1, $h_2\in H$, then $k_1+k_2=\max\{k_1, k_2\}$, $h_1+h_2=\max\{h_1, h_2\}$ and $k_1+h_1=h_1+k_1=\max\{k_1+1, h_1\}$. If $M=\phi$, then it is clear that $(G, M, +)$ is an abelian operator semigroup with an identity. It is also clear that K and $L=\{0, 1\}$ are k-M-subsemigroups of G. For if k_1, $k_1+k_2\in K$, then $k_1+k_2=k_1$ or $k_1+k_2=k_2$. If $k_1+k_2=k_2$, then $k_2\in K$ since $k_1+k_2\in K$. If $k_1+k_2=k_1$ then $k_2\leq k_1$ and it follows that $k_2\in K$. In either case $k_2\in K$ and it follows that K is a k-M-subsemigroup of G. Since G is abelian, it is clear that the series $G\supset K\supset L\supset O$ is a normal k-series of length three. It is also clear that the factors $G/K=\{K, H\}\cong L$ and $K/L=\{K\}\cong O$ are abelian. Consequently, the series is a solvable series and G is solvable. Now define a mapping η: $G\to L$ by $\eta(g)=0$ if $g\in K$ and $\eta(g)=1$ if $g\in H$. Suppose g_1, $g_2\in G$. If g_1, $g_2\in K$, then $g_1+g_2\in K$ and it follows that $\eta(g_1+g_2)=0=0+0=\eta(g_1)+\eta(g_2)$. If g_1, $g_2\in H$, then $g_1+g_2\in H$ and it follows that $\eta(g_1+g_2)=1=1+1=\eta(g_1)+\eta(g_2)$. If $g_1\in K$ and $g_2\in H$, then $g_1+g_2\in H$ and it follows that $\eta(g_1+g_2)=1=0+1=\eta(g_1)+\eta(g_2)$. Consequently, η is a homomorphism such that $\ker \eta=K$. Next, suppose $\eta(g_1)=\eta(g_2)$. Either g_1, $g_2\in K$ or g_1, $g_2\in H$. If g_1, $g_2\in K$ then $g_1+K=\{x\in K|x\geq g_1\}$ and $g_2+K=\{x\in K|x\geq g_2\}$ and it follows that $g_1+\ker\eta\cap g_2+\ker\eta\neq\phi$. If g_1, $g_2\in H$, then a similar argument shows that $g_1+\ker\eta\cap g_2+\ker \eta\neq\phi$. Consequently, η is a semimaximal homomorphism. It is clear that the map $\mu: K\to G$ given $\mu(k)=k$ for all $k\in K$ is a semimaximal homomorphism. Consequently, the sequence

$$O\to K\to G\overset{\mu}{\to}L\overset{\eta}{\to}O$$

is exact since μ and η are semimaximal, μ is one-to-one, and Im $\mu=K=\ker \eta$.

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