Kyungpook Math. J.
Volume 19, Number 2
December, 1979

SOME CHARACTERIZATIONS OF COMPLETELY REGULAR SEMIGROUPS

By S. Lajos

Let S be a semigroup. A subsemigroup A of S is said to be (1,2)-ideal of S if $ASA^2 \subset A$ (cf. [2]). S is said to be *completely regular* if it is the union of disjoint groups. It is known that a semigroup S is completely regular if and only if $a \in aSa^2$ for every element a in S. The author [4] proved that S is completely regular if and only if $ASA^2 = A$ for every (1.2)-ideal A of S. In [5] some further characterizations are given in terms of (1.2)-ideals.

The main result of this short note is contained in the following theorem.

THEOREM 1. *For a semigroup S the following conditions are equivalent*

(A) S is completely regular.

(B) The set 12(S) *of all* (1,2)-*ideals of S is a band with respect to set product.*

(C) The collection 12(S) *of all* (1,2)-*ideals of S is a completely regular semigroup with respect to set product.*

PROOF. (A) implies (B). Let S be a completely regular semigroup, and A, $B \in 12(S)$. First we show that $AB \in 12(S)$. By making use the criterion mentioned above, we obtain $(AB)^2 = ABA(BSB^2) \subset ABSB^2 = AB$, and $(AB)S(AB)^2 = = ABSABA(BSB^2) \subset A(BSB^2) = AB$. Another criterion of the author [5] gives that every (1,2)-ideal of S is globally idempotent. Thus the set $12(S)$ is a band with respect to the ordinary set product.

Evidently (B) implies (C).

Finally, we show that (C) implies (A). Suppose that S is a semigroup with completely regular multiplicative semigroup of (1,2)-ideals. Then, for every element A of $12(S)$, there exists X in $12(S)$ such thta $AXA^2 = A$. Hence it follows that $A = AXA^2 \subset ASA^2 \subset A$. Therefore $ASA^2 = A$ for every (1,2)-ideal A of S, whence S is completely regular (cf. Theorem 2 in [4]).

THEOREM 2. *A semigroup S is completely regular if and only if every principal* (1,2)-*ideal of S has a right identity element.*

PROOF. The necessity follows from Theorem 2 in [5]. Sufficiency. If e is a right identity of $\langle a\rangle$ (1, 2) then we have $ae=a$. On the other hand, $e=a$, a^2, or ata^2 where $t\in S^1$. In each cases it follows that a is completely regular because of $a=ae=a^2$, a^3, or a^2ta^2 with $t\in S^1$. This holds for every element a of S, and thus the semigroup S is completely regular, indeed.

The following theorems can be proved similarly.

THEOREM 3. *A semigroup S is completely regular if and only if every principal* (1, 1)*-ideal of S has a left identity element.*

THEOREM 4. *A semigroup S is the union of groups if and only if every principal* (0, 2)*-ideal of S can be generated by an idempotent element of S.*

COROLLARY. *A regular semigroup S is completely regular if and only if every principal* (0, 2)*-ideal of S is a principal left ideal of S.*

K. Marx University of Economics
Budapest, Hungary

REFERENCES

[1] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. 1, 2nd Edition, Amer. Math. Soc., Providence, R. I., 1964.

[2] S. Lajos, *Generalized ideals in semigroups*, Acta Sci. Math., 22(1961), 217—222.

[3] S. Lajos, *A characterization of Cliffordian semigroups*, Proc. Japan Acad., 52 (1976), 496—497.

[4] S. Lajos, *On the characterization of completely regular semigroups*, Math. Japan. (Special Issue) 20(1975), 33—35.

[5] S. Lajos, (1, 2)*-ideal characterizations of unions of groups*, Math. Seminar Notes (Kobe University) 5(1977), 447—450.