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ON WALLMAN-TYPE EXTENSIONS

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0. Introduction

Topological ordered spaces, namely, topological spaces on which a partial order has defined, were firstly systematically studied by Nachbin [6] and Ward Jr [10]. Since a topological space $(X, \mathscr{T})$ may be considered as a topological ordered space with the discrete partial order, the study of topological ordered spaces not only includes the study of topological spaces, but also reveals many generalization of well-known results concerning topological spaces.

The ordered compactifications of a topological ordered space is comparatively new. In [9], Rodriguez has constructed the order compactification $\beta_o X$ for a completely regular ordered space X, which generalizes Stone-Čech compactification. Moreover, in [3], Hong has introduced the concept of o-completely regular filters on a completely regular ordered space in order to constuct order compactification $\beta_o X$. Recently, in [7], Y.S. Park has introduced the concept of bi-filter on an ordered topological space and constructed an ordered compactification for a convex ordered topological space with semicontinuous order.

In this paper, we introduce the concept of subbasic realcompact topological ordered space, which is a generalization of realcompact space. To do so, we introduce the concept of $(\mathscr{B}_1, \mathscr{B}_2)$-filter in a topological ordered space for some family $\mathscr{B}_1$ and $\mathscr{B}_2$ whose union forms a subbase for the closed subsets. Using this, we define that a topological ordered space is $(\mathscr{B}_1, \mathscr{B}_2)$-realcompact whenever every maximal $(\mathscr{B}_1, \mathscr{B}_2)$-filter with the countable intersection property has nonempty intersection, and that topological ordered space is subbasic realcompact if there are families $\mathscr{B}_1$ and $\mathscr{B}_2$ for which the space is $(\mathscr{B}_1, \mathscr{B}_2)$-realcompact. For a completely regular space X (with the discrete order), it is realcompact if and only if it is $(\mathcal{Z}(X), \mathcal{Z}(X))$-realcompact, where $\mathcal{Z}(X)$ is the set of all zero-sets.

We show that a convex topological ordered space with semicontinuous order is iseomorphic with a dense subspace of a subbasic realcompact pseudo topological ordered space. We also show that every Lindelöf convex topological ordered space is a subbasic realcompact topological ordered space. Finally, we show that, if$(X, \mathscr{T}, \leqq)$ is a convex $(\Gamma_U X, \Gamma_L X)$-realcompact topological ordered space, then $(X, \mathscr{T}, \leqq)$ is iseomorphic with its the subbasic realcompactification.

We deal with the Wallman-type ordered compactifications of a topological ordered space with semicontinous order which are generalizations of Wallman-type compactifications. We introduce the notions of ordered regular subbases and ordered normal subbases.

Whenever there is an ordered regular subbase (resp. ordered normal subbase) on a topological ordered space with semicontinuous order, then one can contruct an ordered compactification of the space. Moreover, it is shown that, if $(X, \mathscr{T}, \leqq)$ is a topological ordered space with semicontinuous order and an ordered normal subbase, then it is a topological ordered space with continuous order and the above compactification is a normally ordered topological space.

For definition and terminology we refer to [1], [2], [4], [5] and [11].

1. Preliminaries

DEFINITION 1.1. A partially ordered set $(X, \leqq)$ is called *discrete* provided that $x \leqq y$ if and only if $x=y$. A map f from a partially ordered set X to a partially ordered set Y is said to be *increasing* (resp. *decreasing*) if $x \leqq y$ in X implies $f(x) \leqq f(y)$ (resp. $f(x) \geqq f(y)$) in Y.

NOTATION and DEFINITION. 1.2. Let $(X, \leqq)$ be a partially ordered set, and A a subset of X. Then we write

$$d(A)=\{y \in X : y \leqq x \text{ for some } x \in A\}$$
$$i(A)=\{y \in X : x \leqq y \text{ for some } x \in A\}$$
$$c(A)=d(A) \cap i(A).$$

In particular, If A is a singleton, say$\{x\}$, then we write $d(X)$ (resp. $i(X)$) instead of $d(\{x\})$ (resp. $i(\{x\})$).

A subset A of X is said to be *decreasing* (resp. *increasing*, resp. *convex*) if $A=d(A)$ (resp. $A=i(A)$, resp. $A=c(A)$).

By a *topological ordered space* we mean a set endowed with both a topology and a partial order.

DEFINITION 1.3. ([10]) Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space, then $\leqq$ is called

(1) *lower semicontinuous* if, whenever $a \not\leqq b$ in X, there exists an open

neighborhood U of a with $x \nleqq b$ for all $x \in U$.

(2) *upper semicontinuous* if for $a \nleqq b$, there exists an open neighborhood V of b with $a \nleqq x$ for all $x \in V$

(3) *semicontinuous* if it is both upper and lower semicontinuous.

(4) *continuous* if, whenever $a \nleqq b$, there exists an open neighborhood U of a and an open neighborhood V of b with $x \nleqq y$ for all $x \in U$ and $y \in V$.

DEFINITION 1.4. ([7]) Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space and let

$$\mathscr{U} = \{U \in \mathscr{T} : U = i(U)\},$$
$$\mathscr{L} = \{U \in \mathscr{T} : U = d(U)\}.$$

Then $\mathscr{U}$ and $\mathscr{L}$ are evidently topologies for X, which are called the *upper* and *lower topology* respectively. Eurthermore, $(X, \mathscr{U})$ is said to be an *upper topological space* and $(X, \mathscr{L})$ a *lower topological space.* We say that a topological ordered space X is *convex* if $\mathscr{L} \cup \mathscr{U}$ is a subbase of the topology of X.

DEFINITION 1.5. Let $(X, \leqq)$ and $(Y, \leqq)$ be partially ordered sets. A map f of $(X, \leqq)$ onto $(Y, \leqq)$ is called an *order isomorphism* if f is one-to-one and $x \leqq y$ if and only if $f(x) \leqq f(y)$.

Let $(X, \mathscr{T}, \leqq)$ and $(Y, \mathscr{T}', \leqq)$ be a topological ordered spaces.

A map f of $(X, \mathscr{T}, \leqq)$ onto $(Y, \mathscr{T}', \leqq)$ is called an *iseomorphism* if it is an order isomorphism and topological homeomorphism.

CONSTRUCTION OF WALLMAN-TYPE COMPACTIFICATION 1.6. ([4]) Let $\mathcal{Z}$ be a normal base for T_1 space X. The Wallman space $\mathscr{W}(\mathcal{Z})$ is obtained in the following way. The points of $\mathscr{W}(\mathcal{Z})$ are the $\mathcal{Z}$-ultrafilters of X. For each A in $\mathcal{Z}$ we define the set A^* to be the family of all $\mathcal{Z}$-ultrafilters having A as a member. The collection of sets A^* for A in $\mathcal{Z}$, is taken as a base for the closed sets of $\mathscr{W}(\mathcal{Z})$. The space $\mathscr{W}(\mathcal{Z})$ is a compact Hausdorff space. There is a natural embedding h of X into $\mathscr{W}(\mathcal{Z})$ where $h(x)$ is the $\mathcal{Z}$-ultrafilter consisting of all $\mathcal{Z}$-sets that contain the element x.

It is well known that the one-point compactification, the Stone-Čech compactification and Fan and Gottesman compactification are Wallman-type compactifications.

DEFINITION 1.7. ([2]) Let $\mathcal{Z}$ be a base for the closed sets in X. Then X is called *$\mathcal{Z}$-realcompact* if every $\mathcal{Z}$-ultrafilter with the countable intersection property has a nonempty intersection.

CONSTRUCTION OF $\mathcal{Z}$-REALCOMPACTIFICATION 1.8. ([2]) Let X be a

Tychonoff space with a countably productive normal base $\mathcal{Z}$ for the closed sets of X. For this base $\mathcal{Z}$, the space $\eta(\mathcal{Z})$ is obtained in the following manner. The points of $\eta(\mathcal{Z})$ are the $\mathcal{Z}$-ultrafilters of X with the countable intersection property. For each Z in $\mathcal{Z}$ we define the set Z^* to be the family of all $\mathcal{Z}$-ultrafilters with the countable intersection property having Z as a member.

The collection $\mathcal{Z}^*$ of sets Z^* for Z in $\mathcal{Z}$ is taken as a base for the closed subsets of $\eta(\mathcal{Z})$.

The space $\eta(\mathcal{Z})$ is a $\mathcal{Z}^*$-realcompact Hausdorff space. There is a natural embedding φ of X into $\eta(\mathcal{Z})$ where $\varphi(x)$ is the $\mathcal{Z}$-ultrafilter consisting of all $\mathcal{Z}$-sets that contain x.

2. Subbasic realcompact ordered spaces

In this chapter, we introduce the concept of subbasic realcompact ordered spaces which is a generalization of that of $\mathcal{Z}$-realcompact spaces introduced by Aló and Shapiro. We show that for any convex topological ordered space with semicotinuous order, it has a subbasic realcompactification, and we investigate properties of those spaces.

In what follows, for a topological ordered space $(X, \mathcal{T}, \leqq)$ we shall write

$\Gamma_U X=\{A\subset X : A$ is a closed decreasing set$\}$, and

$\Gamma_L X=\{A\subset X : A$ is a closed increasing set$\}$.

DEFINITION 2.1. Let $(X, \mathcal{T}, \leqq)$ be a topological ordered space and let $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) be a countably productive subfamily of $\Gamma_U X$ (resp. $\Gamma_L X$). A nonempty subfamily $\mathcal{F}$ (resp. $\mathcal{G}$) of nonempty members of $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) is said to be a *$\mathcal{B}_1$-filter* (resp. *$\mathcal{B}_2$-filter* if $\mathcal{F}$ (resp. $\mathcal{G}$) satisfies the following condition: F_1, $F_2\in\mathcal{F}$ (resp. $\mathcal{G}$) if and only if $F_1\cap F_2\in\mathcal{F}$ (resp. $\mathcal{G}$), i.e., $\mathcal{F}$ (resp. $\mathcal{G}$) is a filter in $\mathcal{B}_1$ (resp. $\mathcal{B}_2$). A pair $(\mathcal{F}, \mathcal{G})$ of $\mathcal{B}_1$-filter $\mathcal{F}$ and $\mathcal{B}_2$-filter $\mathcal{G}$ is said to be a *$(\mathcal{B}_1, \mathcal{B}_2)$-filter* if $F\cap G\neq\phi$ for any $F\in\mathcal{F}$ and $G=\mathcal{G}$. For given two $(\mathcal{B}_1, \mathcal{B}_2)$-filters $(\mathcal{F}_1, \mathcal{G}_1)$ and $(\mathcal{F}_2, \mathcal{G}_2)$ we define a relation $(\mathcal{F}_1, \mathcal{G}_1)\subset(\mathcal{F}_2, \mathcal{G}_2)$ if and only if $\mathcal{F}_1\subset\mathcal{F}_2$ and $\mathcal{G}_1\subset\mathcal{G}_2$.

By a *maximal $(\mathcal{B}_1, \mathcal{B}_2)$-filter* on X we mean a $(\mathcal{B}_1, \mathcal{B}_2)$-filter not contained in any other $(\mathcal{B}_1, \mathcal{B}_2)$-filter under the above relation.

By Zorn's lemma, every $(\mathcal{B}_1, \mathcal{B}_2)$-filter is contained in a maximal $(\mathcal{B}_1, \mathcal{B}_2)$-filter.

DEFINITION 2.2. Let $(X, \mathcal{F}, \leqq)$ be a topological ordered space and let $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) be a countably productive subfamily of $\Gamma_U X$ (resp. $\Gamma_L X$). $(X,$

$\mathscr{T}$, $\leqq$) is called a ($\mathscr{B}_1$, $\mathscr{B}_2$)-*realcompact ordered space* if $\mathscr{B}_1 \cup \mathscr{B}_2$ is a subbase for the closed sets of (X, $\mathscr{T}$) and every maximal ($\mathscr{B}_1$, $\mathscr{B}_2$)-filter ($\mathscr{F}$, $\mathscr{G}$) with the countable intersection property has nonempty intersection, i.e.,

$$\cap\{F : F \in \mathscr{F}\} \cap \{\cap\{G : G \in \mathscr{G}\}\} \neq \phi.$$

(X, $\mathscr{T}$, $\leqq$) is said to be a *subbasic realcompact ordered space* if there exist $\mathscr{B}_1 \subset \Gamma_U X$ and $\mathscr{B}_2 \subset \Gamma_L X$ such that it is a ($\mathscr{B}_1, \mathscr{B}_2$)-realcompat ordered space.

The following two lemmas are immediate from the above definitions.

LEMMA 2.3. *Let* ($\mathscr{F}$, $\mathscr{G}$) *be a maximal* ($\mathscr{B}_1$, $\mathscr{B}_2$)*-filter and* $A \in \mathscr{B}_1$. *Then* $A \in \mathscr{F}$ *if and only if for any* $F \in \mathscr{F}$, $G \in \mathscr{G}$, $A \cap F \cap G \neq \phi$. *The same statement holds for* $\mathscr{G}$.

LEMMA 2.4. *Let* ($\mathscr{F}$, $\mathscr{G}$) *be a maximal* ($\mathscr{B}_1$, $\mathscr{B}_2$)*-filter. Then the following statements are equivalent*:

(1) ($\mathscr{F}$, $\mathscr{G}$) *has the countable intersection property.*

(2) $\mathscr{F}$ *and* $\mathscr{G}$ *are closed under the countable intersection.*

The following proposition is due to Y.S. Park [7].

PROPOSITION 2.5. *Let* ($\mathscr{F}$, $\mathscr{G}$) *be a maximal* ($\mathscr{B}_1$, $\mathscr{B}_2$)*-filter with the countable intersection property. Then the following statement hold*:

Let (A_i) $i \in I$ *be a countable collection in* $\mathscr{B}_1$ (*resp.* $\mathscr{B}_2$). *If* $\cup(A_i) \in \mathscr{F}$ (*resp.* $\mathscr{G}$), *then for some* $i \in I$ *there exists* $A_i \in \mathscr{F}$ (*resp.* $A_i \in \mathscr{G}$).

Let S be an ordered subspace of a topological ordered space (X, $\mathscr{F}$, $\leqq$) and $\mathscr{B}_1$ (resp. $\mathscr{B}_2$) be a countably productive subcollection of $\Gamma_U X$ (resp. $\Gamma_L X$). We write $\mathscr{B}_1(S) = \{B \cap S : B \in \mathscr{B}_1\}$ and $\mathscr{B}_2(S) = \{B \cap S : B \in \mathscr{B}_2\}$.

THEOREM 2.6. *Suppose* $\mathrm{cl}_X Z$ *belongs to* $\mathscr{B}_1$ *for each* $Z \in \mathscr{B}_1(S)$ *and* $\mathrm{cl}_X Y$ *belongs to* $\mathscr{B}_2$ *for each* $Y \in \mathscr{B}_2(S)$. *If* ($\mathscr{B}$, $\mathscr{G}$) *is a maximal* ($\mathscr{B}_1$, $\mathscr{B}_2$)*-filter with the countable intersection property and* $\{F \cap S : F \in \mathscr{F}\}$ *and* $\{G \cap S : G \in \mathscr{G}\}$ *generate a* ($\mathscr{B}_1(S)$, $\mathscr{B}_2(S)$*-filter* ($\mathscr{F}(S)$) *on* S. *Then* ($\mathscr{F}(S)$, $\mathscr{G}(S)$) *is a maximal* ($\mathscr{B}_1(S)$, $\mathscr{B}_2(S)$)*-filter with the countable intersection property.*

PROOF. By the hypothesis ($\mathscr{F}(S)$, $\mathscr{G}(S)$) is a ($\mathscr{F}_1(S)$, $\mathscr{F}_2(S)$)-filter. So there is a maximal ($\mathscr{B}_1(S)$, $\mathscr{B}_2(S)$)-filter ($\mathscr{H}, \mathscr{K}$) containing ($\mathscr{F}(S)$, $\mathscr{G}(S)$). Let $\mathfrak{M} = \{M \in \mathscr{B}_1 :$ there exists $H \in \mathscr{H}$ with $\mathrm{cl}_X H \subset M\}$ and $\mathfrak{N} = \{N \in \mathscr{B}_2 :$ there exists $K \in \mathscr{K}$ with $\mathrm{cl}_X K \subset N\}$.

Then ($\mathfrak{M}$, $\mathfrak{N}$) is a ($\mathscr{B}_1$, $\mathscr{B}_2$)-filter. Since $\mathrm{cl}_X(F \cap S) \subset F$ for $F \in \mathscr{F}$, $F \cap S \in$

$\mathscr{F}(S)$ and $F\in\mathfrak{M}$, Similarly, if $G\in\mathscr{G}$, then $G\in\mathfrak{N}$.

Thus $(\mathscr{F}, \mathscr{G})$ must equal $(\mathfrak{M}, \mathfrak{N})$. By hypothesis, $\mathrm{cl}_X H\in\mathscr{B}_1$ and $\mathrm{cl}_X K\in\mathscr{B}_2$. Hence $\mathrm{cl}_X H\in\mathfrak{M}$, $\mathrm{cl}_X K\in\mathfrak{N}$, $H=\mathrm{cl}_S H=\mathrm{cl}_X H\cap S\in\mathscr{F}(S)$ and $K=\mathrm{cl}_S K=\mathrm{cl}_X K\cap S\in\mathscr{G}(S)$. Therefore $(\mathscr{H}, \mathscr{K})$ is equivalent to $(\mathscr{F}(S), \mathscr{G}(S))$.

Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space with semicontinuous order. For each $x\in X$, we write $\varphi(d(x))=\{A\in\Gamma_U X : d(x)\subset A\}$ and $\varphi(i(x))=\{A\in\Gamma_L X : i(x)\subset A\}$. Then every $\varphi(d(x))$ (resp. $\varphi(i(x))$) is a $\Gamma_U X$-filter (resp. $\Gamma_L X$-filter).

THEOREM 2.7. *Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space with semicontinuous order. Then for each $x\in X$, $(\varphi(d(x)), \varphi(i(x)))$ is a maximal $(\Gamma_U X, \Gamma_L X)$-filter with the countable intersection property.*

PROOF. Let $A\in\varphi(d(x))$ and $B\in\varphi(i(x))$. Then $d(x)\subset A$ and $i(x)\subset B$. Hence $A\cap B\neq\phi$. It follows that $(\varphi(d(x)), \varphi(i(x)))$ is a $(\Gamma_U X, \Gamma_L X)$-filter. Suppose that there exists a $(\Gamma_U X, \Gamma_L X)$-filter $(\mathscr{F}, \mathscr{G})$ such that $(\varphi(d(x)), \varphi(i(x)))\subsetneq(\mathscr{F}, \mathscr{G})$. Then $\varphi(d(x))\subsetneq\mathscr{F}$ or $\varphi(i(x))\subsetneq\mathscr{G}$. Suppose that $\varphi(d(x))\subsetneq\mathscr{F}$. Then there exists a $F\in\mathscr{F}$ such that $F\notin\varphi(d(x))$. Hence $d(x)\not\subset F$ and so $x\notin F$. This implies $i(x)\subset X-F$. Since $i(x)\in\varphi(i(x))$, $i(x)\in\mathscr{G}$. But $i(x)\cap F=\phi$ which is a contradiction. Also in the case that $\phi(i(x))\subsetneq\mathscr{G}$, we obtain a contradiction by the similar argument as the above. Thus $(\varphi(d(x)), \varphi(i(x)))$ is a maximal $(\Gamma_U X, \Gamma_L X)$-filter. It is obvious that it has the countable intersection property.

From now on, we assume that a topological ordered space $(X, \mathscr{T}, \leqq)$ is a topological ordered space with semicontinuous order having a subbsae $\mathscr{B}=\mathscr{B}_1\cup\mathscr{B}_2$ for the closed sets of $(X, \mathscr{T},)$ where $\mathscr{B}_1$ (resp. $\mathscr{B}_2$) is a countably productive subfamily of $\Gamma_U X$ (resp. $\Gamma_L X$).

Let $\mathscr{B}(X)$ denote the collection of all maximal $(\mathscr{B}_1, \mathscr{B}_2)$-filter $(\mathscr{F}, \mathscr{G})$ with the countable intersection property on X. For a given $A\in\mathscr{B}_1$ and $B\in\mathscr{B}_2$, we define the sets $A^d=\{(\mathscr{F}, \mathscr{G})\in\mathscr{B}(X) : A\in\mathscr{F}\}$ and $B^i=\{(\mathscr{F}, \mathscr{G})\in\mathscr{B}(X) : B\in\mathscr{G}\}$ and take the family $\{A^d : A\in\mathscr{B}_1\}\cup\{B^i : B\in\mathscr{B}_2\}$ as a subbase for the closed sets of a topology $\mathscr{T}^*$ on $\mathscr{B}(X)$.

Let us define an order relation $\leqq$ on $\mathscr{B}(X)$ as follows: $(\mathscr{F}_1, \mathscr{G}_1)\leqq(\mathscr{F}_2, \mathscr{G}_2)$ if and only if $\mathscr{F}_2\subset\mathscr{F}_1$ and $\mathscr{G}_1\subset\mathscr{G}_2$ for any $(\mathscr{F}_1, \mathscr{G}_1)$ and $(\mathscr{F}_2, \mathscr{G}_2)$ in $\mathscr{B}(X)$.

Then it is obvious that the relation $\leqq$ is a partial order on $\mathscr{B}(X)$.

PROPOSITION 2.8. *Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space. Define*

${}^{i}U=\{(\mathscr{F},\ \mathscr{G})\in\mathscr{B}(X): F\cap G\cap(X-U)=\phi$ *for some* $F\in\mathscr{F}$ *and* $G\in\mathscr{G}\}$ *and* ${}^{d}L=\{(\mathscr{F},\ \mathscr{G})\in\mathscr{B}(X): F\cap G\cap(X-L)=\phi$ *for some* $F\in\mathscr{F}$ *and* $G\in\mathscr{G}\}$ *where* $U=X-A$ *for some* $A\in\mathscr{B}_1$ *and* $L=X-B$ *for some* $B\in\mathscr{B}_2$. *Then* $(X-U)^d=\mathscr{B}(X)-{}^{i}U$ *and* $(X-L)^i=\mathscr{B}(X)-{}^{d}L$.

PROOF. Suppose that $(\mathscr{F},\ \mathscr{G})\in(X-U)^d$. Then $X-U\in\mathscr{F}$ and so $F\cap G\cap X-U\neq\phi$ for any $F\in\mathscr{F}$ and $G\in\mathscr{G}$. It follows that $(\mathscr{F},\ \mathscr{G})\notin {}^{i}U$, that is, $(\mathscr{F},\ \mathscr{G})\in\mathscr{B}(X)-{}^{i}U$. Conversely, suppose $(\mathscr{F},\ \mathscr{G})\in\mathscr{B}(X)-{}^{i}U$. Then by definition of ${}^{i}U$, $F\cap G\cap(X-U)\neq\phi$ for any $F\in\mathscr{F}$ and $G\in\mathscr{G}$. Therefore $(\mathscr{F},\ \mathscr{G})\in(X-U)^d$. Similarly, $(X-L)^i=\mathscr{B}(X)-{}^{d}L$.

Since basic open sets in $(\mathscr{B}(X),\ \mathscr{T}^*)$ are of form $(\mathscr{B}(X)-A^d)\cap(\mathscr{B}(X)-B^i)$ for some $A\in\mathscr{B}_1$ and $B\in\mathscr{B}_2$. we can obtain the following proposition.

PROPPOSITION 2.9. *Every basic open sets in* $(\mathscr{B}(X),\ \mathscr{T}^*)$ *can be written in the form* ${}^{i}U\cap{}^{d}L$, *where* $X-U$ *and* $X-L$ *are members of* $\mathscr{B}_1$ *and* $\mathscr{B}_2$ *respectively.*

THEOREM. 2.10. *Let* $(X,\mathscr{T},\leqq)$ *be a topological ordered space with a subbase* $\mathscr{B}=\mathscr{B}_1\cup\mathscr{B}_2$ *for the closed sets of* $(X,\mathscr{T})$, *where* $\mathscr{B}_1$ *(resp.* $\mathscr{B}_2$*) is a countably productive subfamily of* $\Gamma_U X$ *(resp.* $\Gamma_L X$*). If* $A\in\mathscr{B}_1$ *and* $B\in\mathscr{B}_2$ *then* A^d *is a closed decreasing set and* B^i *is a closed increasing set in* $\mathscr{B}(X)$. *Moreover,* ${}^{i}(X-A)$ *is an open increasing set and* ${}^{d}(X-B)$ *is an open decreasing set in* $\mathscr{B}(X)$.

PROOF. In order to show that A^d is decreasing, let $(\mathscr{F}_1,\ \mathscr{G}_1)\in A^d$ and $(\mathscr{F}_2,\ \mathscr{G}_2)\leqq(\mathscr{F}_1,\mathscr{G}_2)$ in $\mathscr{B}(X)$. Then $\mathscr{F}_1\subset\mathscr{F}_2$ and since $A\in\mathscr{F}_1$, $A\in\mathscr{F}_2$. Hence $(\mathscr{F}_2,\mathscr{T}_2)\in A^d$. Similarly, B^i is an increasing set. By theorem 2.8, ${}^{i}(X-A)$ is an open increasing set and ${}^{d}(X-B)$ is an open decreasing set in $\mathscr{B}(X)$.

DEFINITION 2.11. Let $(X,\mathscr{T},\leqq)$ be topological ordered space. Then the order $\leqq$ is called *pseudo-continuous* if, whenever $x\nleqq y$ in X, there exists either a neighborhood U of x with $U\nleqq y$, i.e., $z\nleqq y$ for all $z\in U$ or a neighborhood V of y with $x\nleqq V$, i.e., $x\nleqq z$ for all $z\in V$. A topological ordered space with pseudocontinuous partial order is called a ***pseudo topological ordered space.***

THEOREM 2.12. *Let* $(X,\mathscr{T},\leqq)$ *be a topological ordered space. Then the space* $(\mathscr{B}(X),\mathscr{T}^*,\leqq)$ *is a pseudo topological ordered space.*

PROOF. Suppose that $(\mathscr{F}_1,\mathscr{G}_1)\nleqq(\mathscr{F}_2,\mathscr{G}_2)$. Then either $\mathscr{F}_2\not\subset\mathscr{F}_1$ or $\mathscr{G}_1\not\subset\mathscr{G}_2$. If $\mathscr{F}_2\not\subset\mathscr{F}_1$, there exist $F\in\mathscr{F}_2$ and $F\notin\mathscr{F}_1$. It follows that $(\mathscr{F}_1,\mathscr{G}_1)\in\mathscr{B}(X)-F^d$ and $(\mathscr{F}_2,\mathscr{G}_2)\notin\mathscr{B}(X)-F^d$. Therefore $\mathscr{B}(X)-F^d$ is an open neighborhood of $(\mathscr{F}_1, \mathscr{G}_1)$ in $\mathscr{B}(X)$ such that $(\mathscr{B}(X)-F^d) \not\leqq (\mathscr{F}_2, \mathscr{G}_2)$. Similarly, if $\mathscr{G}_1 \not\subset \mathscr{G}_2$, there is an open neighborhood V of $(\mathscr{F}_2, \mathscr{G}_2)$ with $(\mathscr{F}_1, \mathscr{G}_1) \not\leqq V$.

In the following, we assume that $\mathscr{B}_1 = \Gamma_U X$ and $\mathscr{B}_2 = \Gamma_L X$ and $(X, \mathscr{T}, \leqq)$ is a topological ordered space with semicontinuous order.

THOEREM 2.13. *Let $(X, \mathscr{T}, \leqq)$ be a convex topological ordered space. Let us define a map $\varphi : X \longrightarrow \mathscr{B}(X)$ by $\varphi(x) = (\varphi(d(x)), \varphi(i(x)))$ for each $x \in X$. Then φ is a dense embedding from X into $\mathscr{B}(X)$.*

PROOF. To show that φ is one-to-one, let $x \neq y$ in X. Then $x \not\leqq y$ or $y \not\leqq x$. Suppose $x \not\leqq y$. Then $y \notin i(x)$ or $i(y) \not\subset i(x)$. It follows that $i(x) \notin \varphi(i(y))$ or $\varphi(i(x)) \not\leqq \varphi(i(y))$. Hence $\varphi(x) \neq \varphi(y)$. Similarly, if $y \not\leqq x$, then we have $\varphi(x) \neq \varphi(y)$. Thenfore φ is one-to-one. Let $x \leqq y$ in X. Then $y \in i(x)$ and $x \in d(y)$. Since $\varphi(d(y)) \subset \varphi(d(x))$ and $\varphi(i(x)) \subset \varphi(i(y))$, $\varphi(x) \leqq \varphi(y)$. Thus φ is an increasing map. It is obvious that if $\varphi(x) \leqq \varphi(y)$, then $x \leqq y$. This proves that φ is an isomorphism from $(X, \leqq)$ into $(\mathscr{B}(X), \leqq)$. For each $A \in \Gamma_U X$, $A^d \cap \varphi(X) = \{(\varphi(d(x)), \varphi(i(x))) : A \in \varphi(d(x))\} = \{(\varphi(d(x)), \varphi(i(x))) : d(x) \subset A\} = \{\varphi(x) : x \in A\} = \varphi(A)$. Similarly, for each $B \in \Gamma_L X$, $B^i \cap \varphi(X) = \varphi(B)$. Since X is a convex topological ordered space, φ is a homeorphism from X into $\varphi(X)$. Let $\mathscr{B}(X) - (A^d \cup B^i)$ be a nonempty basic open set, where $A \in \Gamma_U X$ and $B \in \Gamma_L X$. Take a maximal $(\Gamma_U X, \Gamma_L X)$-filter $(\mathscr{F}, \mathscr{G}) \in \mathscr{B}(X) - (A^d \cup B^i)$. Then $(\mathscr{F}, \mathscr{G}) \notin A^d$ and $(\mathscr{F}, \mathscr{G}) \notin B^i$. Hence $A \notin \mathscr{F}$ and $B \notin \mathscr{G}$. By proposion 2.3, $A \cup B \neq X$. This implies that $(X-A) \cap (X-B) \neq \phi$. Pick a y in this set, then $\varphi(y) = (\varphi(d(y)), \varphi(i(y))) \in \varphi(X)$. Assume that $\varphi(y) \notin \mathscr{B}(X) - (A^d \cup B^i)$. Then $\varphi(y) \in A^d$ or $\varphi(y) \in B^i$. If $\varphi(y) \in A^d$, $A \in \varphi(d(y))$. Hence $d(y) \subset A$. Thus $y \notin A$ which contradicts the fact that $y \in X-A$. If $\varphi(y) \in B^i$, then we again have a contradiction. Hence $\varphi(y) \in \mathscr{B}(X) - (A^d \cup B^i)$. Thus $\varphi(X) \cap (\mathscr{B}(X) - (A^d \cup B^i)) \neq \phi$, that is, $\varphi(X)$ is a dense subset of $\mathscr{B}(X)$.

THEOREM 2.14. *Let $(X, \mathscr{T}, \leqq)$ be a convex topological ordered space and let φ be the natural embedding of X into $\mathscr{B}(X)$. If A, B and $(B_n)_{n=1}^{\infty}$ are members of $\Gamma_U X$ and $(U_n)_{n=1}^{\infty}$ are members of $\mathscr{U} = \{U : X-U \in \Gamma_U X\}$, then the following properties hold*:

(1) *If $A \subset B$, then $A^d \subset B^d$.*
(2) ${}^i(\bigcup_{n=1}^{\infty} U_n) = \bigcup_{n=1}^{\infty} {}^iU_n$ *and* ${}^i(\bigcap_{n=1}^{\infty} U_n) = \bigcap_{n=1}^{\infty} {}^iU_n$.
(3) ${}^iU \cap \varphi(X) = \varphi(U)$.
(4) $\mathrm{cl}_{\mathscr{B}(X)}\, \varphi(B) = B^d$.

(5) $(\bigcap_{n=1}^{\infty} B_n)^d = \bigcap_{n=1}^{\infty} B_n^d$ *and* $(\bigcup_{n=1}^{\infty} B_n)^d = \bigcup_{n=1}^{\infty} B_n^d$.

If A, B *and* $(B_n)_{n=1}^{\infty}$ *are members of* $\Gamma_L X$ *and* U, $(U_u)_{n=1}^{\infty}$ *are members of* $\mathcal{L} = \{U : X - U \in \Gamma_L X\}$, *then we have the similar results.*

PROOF. (1) If $(\mathcal{F}, \mathcal{G}) \in A^d$, then $A \in \mathcal{F}$, and since $A \subset B$, $B \in \mathcal{F}$. Thus $(\mathcal{F}, \mathcal{G}) \in B^d$.

(2) If $(\mathcal{F}, \mathcal{G}) \in \bigcup_{n=1}^{\infty} {}^iU_n$, then $(\mathcal{F}, \mathcal{G}) \in {}^iU_n$ for some n, i.e., $F \cap G \cap (X - U_n) = \phi$ for some $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Hence $F \cap G \cap (X - \bigcup U_n) = \phi$ and so $(\mathcal{F}, \mathcal{G}) \in {}^i(\bigcup U_n)$. Conversely, if $(\mathcal{F}, \mathcal{G}) \in {}^i(\bigcup_{n=1}^{\infty} U_n)$, then $F \cap G \cap (X - \bigcup U_n) = \phi$ for some $F \in \mathcal{F}$ and $G \in \mathcal{G}$. If $(\mathcal{F}, \mathcal{G}) \notin \bigcup_{n=1}^{\infty} {}^iU_n$, then $(\mathcal{F}, \mathcal{G}) \notin {}^iU_n$, that is, $(\mathcal{F}, \mathcal{G}) \in \mathcal{B}(X) - {}^iU_n$ for each n. Since $\mathcal{B}(X) - {}^iU_n = (X - U_n)^d$, $(\mathcal{F}, \mathcal{G}) \in (X - U_n)^d$ for each n. It follows that $X - U_n \in \mathcal{F}$ for each n. By lemma 2.3, for each n, $F \cap G \cap (X - U_n) \neq \phi$ for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Since $(\mathcal{F}, \mathcal{G})$ has the countable intersection property, $F \cap G \cap (\bigcap_{n=1}^{\infty} (X - U_n)) \neq \phi$ for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$. This contradiction establishes that if $(\mathcal{F}, \mathcal{G}) \in {}^i(\bigcup_{n=1}^{\infty} U_n)$, then $(\mathcal{F}, \mathcal{G}) \in \bigcup_{n=1}^{\infty} {}^iU_n$. Using this and $\mathcal{B}(X) - {}^iU = (X - U)^d$ for $(X - U) \in \Gamma_U X$. We have ${}^i(\bigcap_{n=1}^{\infty} U_n) = \bigcap_{n=1}^{\infty} {}^iU_n$.

(3) Since $B^d \cap \varphi(X) = \varphi(B)$ and φ is a homeomorphism, $\varphi(U) = {}^iU \cap \varphi(X)$.

(4) Since $B^d \cap \varphi(X) = \varphi(B)$, $\mathrm{cl}_{\mathcal{B}(X)}\, \varphi(B) \subset B^d$. Conversely, let $(\mathcal{F}, \mathcal{G}) \in B^d$ and ${}^iU \cap {}^dL$ be any basic open set containing $(\mathcal{F}, \mathcal{G})$. Then $F_1 \cap G_1 \cap (X - U) = \phi$ for some $F_1 \in \mathcal{F}$ and $G_1 \in \mathcal{G}$ and $F_2 \cap G_2 \cap (X - L) = \phi$ for some $F_2 \in \mathcal{F}$ and $G_2 \in \mathcal{G}$. Also $B \in \mathcal{F}$ since $(\mathcal{F}, \mathcal{G}) \in B^d$. By lemma 2.3, $F_1 \cap F_2 \cap G_1 \cap G_2 \cap B \neq \phi$. Let $x \in F_1 \cap F_2 \cap G_1 \cap G_2 \cap B$. Then $\varphi(x) \in {}^iU \cap {}^dL$. For, $d(x) \subset F_1$, $d(x) \subset F_2$, $i(x) \subset G_1$, and $i(x) \subset G_2$. It follows that F_1, $F_2 \in \varphi(d(x))$ and G_1, $G_2 \in \varphi(i(x))$. Since $F_1 \cap G_1 \cap (X - U) = \phi$ and $F_2 \cap G_2 \cap (X - L) = \phi$, Therefore $\varphi(x) \in {}^iU \cap {}^dL \cap \varphi(B)$.

(5) $(\bigcap B_n)^d = \mathcal{B}(X) - {}^i(X - \bigcap B_n) = \mathcal{B}(X) - {}^i(\bigcup (X - B_n)) = \mathcal{B}(X) - (\bigcup {}^i(X - B_n)) = \bigcap (\mathcal{B}(X) - {}^i(X - B_n)) = \bigcap B_n^d$. Similarly, $(\bigcup_{n=1}^{\infty} B_n)^d = \bigcup_{n=1}^{\infty} B_n^d$.

THEOREM 2.15. *Let* $(A_n)_{n=1}^{\infty}$ *be a sequence in* $\mathcal{B}_1$ *and* $(B_n)_{n=1}^{\infty}$ *a sequence in* $\mathcal{B}_2$. *Then* $(\bigcap A_n^d) \cap (\bigcap B_n^i) = \phi$ *if and only if* $(\bigcap A_n) \cap (\bigcap B_n) = \phi$. *Moreover,* $(\bigcap A_n)^d \cap (\bigcap B_n)^i = \mathrm{cl}\varphi((\bigcap A_n) \cap (\bigcap B_n))$.

PROOF. Suppose $(\bigcap A_n)^d \cap (\bigcap B_n)^i = \phi$. $\mathrm{cl}_{\mathcal{B}(X)} \varphi((\bigcap A_n) \cap (\bigcap B_n)) \subset \mathrm{cl}_{\mathcal{B}(X)}\, \varphi(\bigcap A_n) \cap \mathrm{cl}_{\mathcal{B}(X)} \varphi(\bigcap B_n) = (\bigcap A_n)^d \cap (\bigcap B_n)^i$. It follows that $(\bigcap A_n) \cap (\bigcap B_n) = \phi$. Conversely, If $(\bigcap A_n^d) \cap (\bigcap B_n^i) \neq \phi$, then we can take a $(\mathcal{F}, \mathcal{G})$ in $(\bigcap A_n^d) \cap (\bigcap B_n^i)$. Hence $(\mathcal{F}, \mathcal{G}) \in \bigcap A_n^d$ and $(\mathcal{F}, \mathcal{G}) \in \bigcap B_n^i$, i.e., $A_n \in \mathcal{F}$ for all n and $B_n \in \mathcal{G}$ for all n. Since $(\mathcal{F}, \mathcal{G})$ is a maximal $(\mathcal{B}_1, \mathcal{B}_2)$- filter with the countable intersection property, $(\cap A_n)\cap(\cap B_n)\neq\phi$. It will suffices to show that $(\cap A_n)^d\cap(\cap B_n)^i\subset$ cl $\varphi((\cap A_n)\cap(\cap B_n))$. Let $(\mathscr{F}, \mathscr{G})\in(\cap A_n)^d\cap(\cap B_n)^i$ and ${}^iU\cap{}^dL$ be any basic open set containing $(\mathscr{F},\mathscr{G})$. Then $\cap A_n\in\mathscr{F}$, $\cap B_n\in\mathscr{G}$, $F_1\cap G_1\cap(X-U)=\phi$ for some $F_1\in\mathscr{F}$ and $G_1\in\mathscr{G}$ and $F_2\cap G_2\cap(X-L)=\phi$ for some $F_2\in\mathscr{F}$ and $G_2\in\mathscr{G}$. Since $(\mathscr{F},\mathscr{G})$ is a $(\mathscr{B}_1,\mathscr{B}_2)$-filter with the countable intersection property, $M=F_1\cap F_2\cap(\cap A_n)\cap G_1\cap G_2\cap(\cap B_n)\neq\phi$. Suppose $x\in M$. Then $\varphi(x)\in{}^iU\cap{}^dL\cap\varphi((\cap A_n)\cap(\cap B_n))$.

For $\mathscr{B}=\Gamma_UX\cap\Gamma_LX$, we write $\Gamma_U{}^*\mathscr{B}(X)=\{F^d : F\in\Gamma_UX\}$. $\Gamma_L{}^*\mathscr{B}(X)=\{G^i : G\in\Gamma_LX\}$ and $\mathscr{B}^*=\Gamma_U{}^*\mathscr{B}(X)\cup\Gamma_L{}^*\mathscr{B}(X)$.

THEOREM 2.16. *If $(X,\mathscr{T},\leqq)$ is a convex topological ordered space. Then $(X,\mathscr{T},\leqq)$ is iseomorphic to a dense subspace of a subbasic realcompact pseudo topological ordered space $(\mathscr{B}(X),\mathscr{T}^*,\leqq)$.*

PROOF. We may assume that $\varphi: X\to\mathscr{B}(X)$ is the natural embeding, that is, we identify x with $\varphi(x)$ for all $x\in X$: hence X may be considered as a subspace of $\mathscr{B}(X)$. By theorem 2.12 and 2.13, it is enough to show that every maximal $(\Gamma_U{}^*\mathscr{B}(X), \Gamma_L{}^*\mathscr{B}(X))$-filter with the countable intersection property has nonempty interesection. Let (α,β) be a maximal $(\Gamma_U{}^*\mathscr{B}(X), \Gamma_L{}^*\mathscr{B}(X))$-filter on $\mathscr{B}(X)$ with the countable intersection property and let $\mathscr{F}=\{F\in\Gamma_UX : F^d\in\alpha\}=\{F^d\cap X : F^d\in\alpha\}$ and $\mathscr{G}=\{G\in\Gamma_LX : G^i\in\beta\}=\{G^i\cap X : G^i\in\beta\}$. Then by theorem 2.15, $(\mathscr{F},\mathscr{G})$ is a (Γ_UX, Γ_LX)-filter. Since (α,β) has the countable intersection property, $(\mathscr{F},\mathscr{G})$ has also the countable intersection property. By theorem 2.6, $(\mathscr{F},\mathscr{G})$ is a maximal (Γ_UX, Γ_LX)-filter with the countable intersection property on X and $(\mathscr{F},\mathscr{G})\in(\cap\alpha)\cap(\cap\beta)$.

THEOREM 2.17. *Every Lindelöf convex topological ordered space is a subbasic realcompact ordered space.*

PROOF. Let $(X,\mathscr{T},\leqq)$ be a Lindelöf convex topological ordered space. Since it is convex, $\Gamma_UX\cup\Gamma_LX$ is a subbase for the closed sets and hence it is enough to show that $(X,\mathscr{T},\leqq)$ is a (Γ_UX, Γ_LX)-realcompact. Let $(\mathscr{F},\mathscr{G})$ be a maximal (Γ_UX, Γ_LX)-filter with the countable intersection property such that it has empty intersection, i.e., $\cap(\mathscr{F}\cap\mathscr{G})=\phi$. Then $\{X-F : F\in\mathscr{F}\}\cup\{X-G : G\in\mathscr{G}\}=X$. Since $(X,\mathscr{T},\leqq)$ is a Lindelöf ordered space, there is a countable subfamily (F_n) of $\mathscr{F}$ and (G_n) of $\mathscr{G}$ such that $\{\cup_{n=1}^{\infty}(X-F_n)\}\cup\{\cup_{n=1}^{\infty}(X-G_n)\}=X$. It follows that $(\cap_{n=1}^{\infty}F_n)\cap(\cap_{n=1}^{\infty}G_n)=\phi$ which is a contradiction.

Since the real line R is a Lindelöf space and $\mathscr{B}=\{[x,\to] : x\in R\}\cap\{[\leftarrow,x] :$

$x \in R\}$ is a subbase for the closed sets, we have

COROLLARY 2.18. *Let* $(R, \mathcal{T}, \leqq)$ *be the real line equipped with the usual topology and usual order. Then it is a subbasic realcompact ordered space.*

THEOREM 2.19. *Let* $(X, \mathcal{T}, \leqq)$ *be a convex* $(\Gamma_U X, \Gamma_L X)$*-realcompact ordered space. Then* $(X, \mathcal{T}, \leqq)$ *is iseomorphic with* $(\mathscr{B}(X), \mathcal{T}^*, \leqq)$.

PROOF. Let $(\mathscr{F}, \mathcal{T})$ be a maximal $(\Gamma_U X, \Gamma_L X)$-filter with the countable intersection property on X. Since X is a $(\Gamma_U X, \Gamma_L X)$-realcompact ordered space, we can pick x in $\bigcap(\mathscr{F} \cup \mathscr{G})$. If $F \in \mathscr{F}$, $d(x) \subset F$. Thus $F \in \varphi(d(x))$. Similarly, if $G \in \mathscr{G}$, then $G \in \varphi(i(x))$. Hence $(\mathscr{F}, \mathscr{G}) \subset (\varphi(d(x)), \varphi(i(x)))$. By the maximality of $(\mathscr{F}, \mathscr{G})$, $(\mathscr{F}, \mathscr{G})$ must equal to $(\varphi(d(x)), \varphi(i(x)))$. It follows that $\varphi(X) = \mathscr{B}(X)$.

For a completely regular space (with the discrete order), it is realcompact if and only if it is $(\mathcal{Z}(X), \mathcal{Z}(X))$-realcompact, where $\mathcal{Z}(X)$ is the set of all zoro-sets.

3. Wallman-type ordered compactifications

The Wallman base concept, first initiated by H. Wallman in 1938, has been successfully used by Orrin Frink [4]. He has introduced the notion of a normal base for the closed subsets of a space. For a normal base $\mathcal{Z}$ of a completely regular T_1 space X, he has constructed the Wallman space $W(\mathcal{Z})$ consisting of the $\mathcal{Z}$-ultrafilters, which is a compactification of X.

In this chapter, we introduce the concept of an ordered regular subbase (resp. ordered normal subbase) for a topological ordered space to obtain the compact pseudo topological ordered space (resp. compact topological ordered space with continuous order) which is an ordered compactification of the space.

We assume all our topological ordered spaces to be topological ordered spaces with semicontinuous order and we shall use subbases for the closed sets instead of open sets.

DEFINITION 3.1. Let $(X, \mathcal{T}, \leqq)$ be a topological ordered space with semicontinuous order and S_0 be a subfamily of $\Gamma_U X \cup \Gamma_L X$. The family S_0 is called an *ordered regular subbase* for X if S_0 satisfies the following conditions:

(1) S_0 is a subbase for the closed sets of $(X, \mathcal{T})$.

(2) For any closed decreasing (resp. increasing) set A and any point x not

in A, there exists a closed increasing (resp. decreasing) set B in S_0 such that $A\cap B=\phi$ and $x\in B$.

Let $(R, \mathscr{T}, \leqq)$ be the real line with usual topology and usual order. Then $\Gamma_U R\cup\Gamma_L R$ is an ordered regular subbase for R.

We note that a nonempty family α of sets is said to have the finite intersection property if the intersection of any finite subfamily of α is nonempty.

DEFINITION 3.2. Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space with an ordered regular subbase S_0. A nonempty subfamily ξ of S_0 is called an *S_0-centered system* on X if it has the finite intersection property. By a *maximal S_0-centered system* on X we mean an S_0-centered system which is maximal with respect to this property.

By Zorn's lemma, every S_0-centered system is contained in a maximal S_0-centered system.

LEMMA 3.3. *Let S_0 be an ordered regular subbase for a topological ordered space $(X, \mathscr{T}, \leqq)$. For each $x\in X$, we define $\varphi(x)=\{A\in S_0 : x\in A\}$. Then $\varphi(x)$ is a maximal S_0-centered system on X.*

PROOF. By the definition of $\varphi(x)$, it is an S_0-centered system on X. Now suppose ξ is an S_0-centered system on X such that $\varphi(x)\subsetneqq\xi$. Then there exists an $A\in\xi$ with $x\notin A$. Suppose A is a closed decreasing set. Then there is an increasing set B in S_0 such that $A\cap B=\phi$ and $x\in B$.

Since $\varphi(x)\subset\xi$ and $B\in\varphi(x)$, $B\in\xi$. Hence $A\cap B\neq\phi$ which is a contradiction. Similarly, if A is a closed increasing set, then we have a contradiction which assures the maximality of $\varphi(x)$.

PROPOSITION 3.4. *Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space with an ordered regular subbase S_0 and let ξ be a maximal S_0-centered system on X. If $A, B\in S_0$ and $A\cup B=X$, then $A\in\xi$ or $B\in\xi$.*

PROOF. Suppose that $A\notin\xi$ and $B\notin\xi$. Then there are $T_j\in\xi$, $j=1, 2, \cdots, m$, such that $A\cap(\cap\{T_j : j=1, \cdots, m\})=\phi$ and $S_k\in\xi$, $k=i, \cdots, n$, such that $B\cap(\cap\{S_k : k=1, \cdots, n\})=\phi$. Since ξ is an S_0-centered system, $(\cap\{T_j : j=1, \cdots, m\})\cap(\cap\{S_k : k=1, \cdots, n\})\neq\phi$. We can take x from this set; therefore $x\in A$ or $x\in B$. Thus $A\cap(\cap\{T_j : j=1, \cdots, m\})\neq\phi$ or $B\cap(\cap\{S_k : k=1, \cdots, n\})\neq\phi$. This completes the proof.

LEMMA 3.5. *Let ξ be a maximal S_0-centered system and let $A, B\in S_0$ with*

$A\cap B\in S_0$. *Then* $A, B\in\xi$ *if and only if* $A\cap B\in\xi$.

PROOF. $(\leftarrow)$: obvious.

$(\rightarrow)$: Suppose $A, B\in\xi$. Then $\xi\cup\{A\cup B\}$ is an S_0-centered system and $\xi\subset\xi\cup\{A\cap B\}$. It follows that $A\cap B\in\xi$.

Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space. We denote by $W(S_0)$ the set of all maximal S_0-centered system in S_0. For each $A\in S_0$, we define the set $A^*=\{\xi\in W(S_0) : A\in\xi\}$.

$W(S_0)$ will be endowed with a topology $\mathscr{T}^*$ for which the family $\{A^* : A\in S_0\}$ serves as a subbase for the closed sets. An order relation $\leqq$ on $W(S_0)$ is defined as follows: $\xi\leqq\mathfrak{N}$ in $W(S_0)$ if and only if $\mathfrak{N}\cap\Gamma_U X\subset\xi$ and $\xi\cap\Gamma_L X\subset\mathfrak{N}$. The subbasic open sets of $W(S_0)$ can be identified taking complements of the subbasic closed sets:

$$\begin{aligned} W(S_0)-A^* &= \{\xi\in W(S_0) : A\notin\xi\} \\ &= \{\xi\in W(S_0) : \text{ there exist } B_j\in\xi \ (j=1, \cdots, n) \text{ with } \\ &\quad A\cap(\cap\{B_j : j=1, \cdots, n\})=\phi\}. \\ &= \{\xi\in W(S_0) : \text{there exist } B_j\in\xi \ (j=1, \cdots, n) \text{ with } \\ &\quad \cap\{B_j : j=1\cdots, n\}\subset X-A\}. \end{aligned}$$

For $U=X-A$, denote the subbasic open set obtained from U by $^*U=\{\xi\in W(S_0) :$ there exist finite members B_j of ξ such that $\cap B_j\subset U\}$, i.e. $^*U=W(S_0)-(X-U)^*$.

LEMMA 3.6. *If A is a decreasing (resp. increasing) set in ordered regular subbase S_0, then A^* is a closed decreasing (resp. increasing) set in $W(S_0)$.*

PROOF. Suppose $\xi\in A^*$ and $\mathfrak{N}\leqq\xi$. Then if $A\in\Gamma_U X$ and $A\in\xi$, then $A\in\mathfrak{N}$ by the definition of $\leqq$ in $W(S_0)$. Hence $\mathfrak{N}\in A^*$. Similarly, let $A\in\Gamma_L X$. Suppose $\xi\in A^*$ and $\xi\leqq\mathfrak{N}$. Then $A\in\xi$ and $A\in\mathfrak{N}$. Thus $\mathfrak{N}\in A^*$.

Let us define a map $\varphi : X\longrightarrow W(S_0)$ by $\varphi(x)=\{A\in S_0 : x\in A\}$ for each $x\in X$, where S_0 is an ordered regular subbase for a topological ordered space X.

THEOREM 3.7. *If S_0 is an ordered regular subbase for a topological ordered space $(X, \mathscr{T}, \leqq)$, then the following properties hold.*

(1) $\varphi(A)=A^*\cap\varphi(X)$ *for each A in S_0.*

(2) $\mathrm{cl}_{W(S_0)}\varphi(A)=A^*$ *for eaah A in S_0.*

(3) *If $A\subset B$, then $A^*\subset B^*$ for A, B in S_0.*

PROOF. (1) If A is a member of S_0, then $A^*\cap\varphi(X)=\{\varphi(x) : A\in\varphi(x)\}=\{\varphi(x)$

$: x \in A\} = \varphi(A)$.

(2) From (1), it follows that $\mathrm{cl}_{W(S_0)}\varphi(A)$ is contained in A^*. Let $\xi \in A^*$ and $\mathscr{U}$ be basic open set containing ξ. Then $A \in \xi$ and $\mathscr{U}$ is the finite intersection of subbasic open sets $^*V_j$. For each j, there exist finite members B_{jk} of ξ with $\cap B_{jk} \subset V_j$. Thus $A \cap (\cap \{B_{jk}\}) \neq \phi$. Pick x from this set, then $\varphi(x) \in \varphi(A) \cap \mathscr{U}$. This proves (2).

(3) By maximality of ξ, if $A \in \xi$ and $A \subset B$, then $B \in \xi$.

By proposition 3.4, we have the following lemma.

LEMMA 3.8. *Let $A_i \in S_0$ $(i=1, \cdots, n)$ and $\cup \{A_i : i=1, \cdots, n\} = X$. Then $\cup \{A_i^* : i=1, \cdots, n\} = W(S_0)$.*

The following is also immediate.

LEMMA 3.9. *Let $A_i \in S_0$, $i=1, \cdots, n$. Then $\cap \{A_i : i=1, \cdots, n\} = \phi$ if and only if $\cap \{A_i^* : i=1, \cdots, n.\} = \phi$.*

LEMMA 3.10. *Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space with an ordered regular subbase S_0. Then the map φ is a dense embedding into $W(S_0)$.*

PROOF. Let $x \neq y$ in X. Then $x \not\leqq y$ or $y \not\leqq x$. If $x \not\leqq y$, then $x \notin d(y)$ *and* $y \in d(y)$. Since $x \notin d(y)$, there exist a closed increasing set B in S_0 such that $B \cap d(y) = \phi$ and $x \in B$. This implies $y \notin B$ and $x \in B$. It follows that $B \notin \varphi(y)$ and $B \in \varphi(x)$, i.e., $\varphi(x) \not\leqq \varphi(y)$. Hence $\varphi(x) \neq \varphi(y)$. Similarly, if $y \not\leqq x$, then $\varphi(x) \neq \varphi(y)$. This proves that φ is one-to-one. To show that φ is an increasing map, let $x \leqq y$ in X. If A is a closed decreasing set in $\varphi(y)$, then $y \in A$ and $x \in A$ since $x \leqq y$. Therefore $A \in \varphi(x)$. Suppose B is a closed increasing set in $\varphi(x)$. Then $B \in \varphi(y)$ by a similar argument as the above. This proves that $\varphi(x) \leqq \varphi(y)$. By the fact that if $x \not\leqq y$, then $\varphi(x) \not\leqq \varphi(y)$, if $\varphi(x) \leqq \varphi(y)$, then $x \leqq y$. Hence φ is an isomorphism from $(X, \leqq)$ into $(W(S_0), \leqq)$. Since $A^* \cap \varphi(X) = \varphi(A)$ for $A \in S_0$ by theorem 3.7, φ is a homeomorphism from $(X, \mathscr{T})$ onto $\varphi(X)$. Let $W(S_0) - (A_1^* \cup \cdots \cup A_n^*)$ be a nonempty basic open set in $W(S_0)$. Then by lemma 3.8, $A_1 \cup \cdots \cup A_n \neq X$. This implies that $(X - A_1) \cup \cdots \cup (X - A_n) \neq \phi$. We can take x in this set. Since $x \notin A_j$ for all $j = i, \cdots, n$, $\varphi(x) \notin A_1^* \cup \cdots \cup A_n^*$. Thus $\varphi(x) \in (W(S_0) - (A_1^* \cup \cdots \cup A_n^*)) \cap \varphi(X)$. This proves that $\varphi(X)$ is dense in $W(S_0)$.

THEOREM 3.11. *Let $(X, \mathscr{T}, \leqq)$ be a topological ordered space with semicontinuous order and an ordered regular subbase S_0. Then $(W(S_0), \mathscr{T}^*, \leqq)$ is a* *compact pseudo topological ordered space and* $\varphi:(X,\mathscr{T},\leqq)\longrightarrow(W(S_0),\mathscr{T}^*,\leqq)$ *is a dense embedding.*

PROOF. Suppose $\xi\not\leqq\eta$. Then either there exists a closed decreasing set $A\in\eta$ and $A\notin\xi$ or a closed increasing set $B\in\xi$ and $B\notin\eta$. If there is a closed decreasing set $A\in\eta$ and $A\notin\xi$, then $\xi\in W(S_0)-A^*$ and $W(S_0)-A^*\not\leqq\eta$. Similarly, suppose there exists a closed increasing set $B\in\xi$ and $B\notin\eta$. Then $\eta\in W(S_0)-B^*$ and $\xi\leqq W(S_0)-B^*$. Therefore $W(S_0)$ is a pseudo topological ordered space. In order to prove that the compactness of $W(S_0)$, it suffices to show that each family of subbasic closed sets having a finite intersection property has nonempty intersection. Let $\mathscr{D}^*=\{A_i^*:A_i\in S_0\}$ be a family of subbasic closed sets having a finite intersection property on $W(S_0)$. Then $\mathscr{D}=\{A_i:A_i^*\in\mathscr{D}^*\}$ has the finite intersection property by lemma 3.9 and is contained in a maximal S_0-centered system ξ. But since $A_i\in\xi$ for all $A_i\in\mathscr{D}$, $\xi\in\cap\{A_i^*:A^*\in\mathscr{D}^*\}$.

Now we introduce the concept of an ordered normal subbase in order to obtain an ordered compactifications with continuous order.

DEFINITION 3.12. Let $(X,\mathscr{T},\leqq)$ be a topological ordered space with semicontinuous order and let S_0 be a subfamily of $\Gamma_U X\cup\Gamma_L X$. The family S_0 is called an *ordered normal subbase* for X if S_0 satisfies the following conditions:

(1) S_0 is an ordered regular subbase for X.

(2) If $A\cap(\bigcap_{j=1}^{n}B_j)=\phi$, where A is a decreasing (resp. increasing) set in S_0 and $B_j(j=1,\cdots,n)$ in S_0, then there exist decreasing (resp. increasing) set C and increasing (resp. decreasing) set D in S_0 such that $A\subset X-D$, $\bigcap_{j=1}^{r}B_j\subset X-C$ and $(X-C)\cap(X-D)=\phi$.

The family $\Gamma_U R\cup\Gamma_L R$ is also an ordered normal subbase for the real line R with the usual topology and usual order.

Since an ordered normal subbase S_0 on a topological ordered space $(X,\mathscr{T},\leqq)$ with semicontinuous order is, by definition, a regular ordered subbase on $(X,\mathscr{T},\leqq)$, by theorem 3.11 one can construct a compact pseudo topological ordered space $(W(S_0),\mathscr{T}^*,\leqq)$ and a dense embedding $\varphi:(X,\mathscr{T},\leqq)\longrightarrow(W(S_0),\mathscr{T}^*,\leqq)$. Moreover. for any decreasing set C and increasing set D in S_0, $*(X-C)=W(S_0)-C^*$ is an increasing open set and $*(X-D)=W(S_0)-D^*$ is a decreasing open set.

LEMMA 3.14. *Let* $(X, \mathscr{T}, \leqq)$ *be a topological ordered space with an ordered normal subbase* S_0. *Then* $(W(S_0), \mathscr{T}^*, \leqq)$ *is a compact topological ordered space with continuous order.*

PROOF. It suffices to show that the order $\leqq$ is continuous. Suppose $\xi \leqq \eta$ in $W(S_0)$. Then there exists a decreasing set $A \in \eta$ such that $A \cup (\cup B_j) = \phi$ for some $B_j (j=1, \cdots, n)$ in ξ or there exists an increasing set $B \in \xi$ such that $B \cap (\cap A_j) = \phi$ for some $A_j (j=1, \cdots, n)$ in η. Suppose there is a decreasing set $A \in \eta$ such that $A \cap (\cap B_j) = \phi$ for some $B_j (j=1, \cdots, n)$ in ξ. Then we have a decreasing set C and increasing set D in S_0 such that $A \subset X - D$, $\cap B_j \subset X - C$ and $(X - C) \cap (X - D) = \phi$. Let $X - D = U$ and $X - C = V$. Then $\eta \in {}^*U$, $\xi \in {}^*V$, and since $U \cap V = \phi$, ${}^*U \cap {}^*V = \phi$. ${}^*U$ is a decreasing open neighborhood of η and ${}^*V$ is an increasing open neighborhood of ξ. Similarly, one can show the similar fact for the case of $B \in \xi$ and $(\cap A_j) \cap B = \phi$. This proves that the order $\leqq$ is continuous.

By the lemma 3.10, 3.11 and 3.14, we have the following theorem.

THEOREM 3.15. *Let* $(X, \mathscr{T}, \leqq)$ *be a topological ordered space with semicontinuous order having an ordered normal subbase* S_0. *Then* $(W(S_0), \mathscr{T}^*, \leqq)$ *is a compact topological ordered space with continuous order in which X is densely embedded.*

The following is immediate from the above theorem.

COROLLARY 3.16. *If* $(X, \mathscr{T}, \leqq)$ *is a topological ordered space with semicontinuous order and it has an ordered normal subbase, then it is a topological ordered space with continuous order.*

Since every compact topological ordered space with continuous order is normally ordered, we have

THEOREM 3.17. *Let* $(X, \mathscr{T}, \leqq)$ *be a topological ordered space with semicontinuous order having an ordered normal subbase. Then* $(W(S_0), \mathscr{T}^*, \leqq)$ *is a normally ordered space.*

We note that if the given order on topological ordered space $(X, \mathscr{T}, \leqq)$ is discrete, then the theorem reduces to the Wallman-type compactification of topological space.

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