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A GENERALISATION OF CERTAIN RESULTS OF G. JAMESON

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1. Introduction

G. J. O. Jameson in his paper "Convex Series" published in Proc. Camb. Phil. Soc. 72, 37—47, (1972), has studied the CS-closed and CS-compact subsets of a Hausdorff topological linear space. In the present work, we introduce the concepts of A-p-CS-closed and A-p-CS-compact subset of a Hausdorff topological vector space over the field K of real or complex numbers, finding, in a natural way, certain results concerning locally bounded topological vector spaces: for instance, the closed-graph and open-mapping theorems.

2. Definitions and notations

Let A be a subset of a Hausdorff topological vector space and p a real number such that $0<p\leq 1$. A series of the form $\sum_1^\infty \lambda_n a_n$, where $a_n \in A$ and $\lambda_n \in K$, $n\in N$ and $\sum_1^\infty |\lambda_n|^p \leq 1$ is called an *absolutely p-convex series of elements of A*.

We say that A is:

a) *A-p-CS-closed* if it contains the sum of every convergent absolutely p-convex series of its elements.

b) *A-p-CS-compact* if every absolutely p-convex series of its elements converges to a point of A.

The absolutely p-convex cover of a set A will be denoted by A-p-$co(A)$, the image of A under a relation R by $R(A)$ and the interior of A by $I(A)$. A set admitting addition and non-negative scalar multiplication will be called a *wedge*.

3. Preliminaries and preservation rules

Every A-p-CS-compact is A-p-CS-closed and every A-p-CS-closed is A-p-convex.

If each member of a family of sets is A-p-CS-closed (or A-p-CS-compact), then so is their intersection.

A linear subspace is A-p-CS-closed if and only if it is sequentially closed.

Every sequentially closed A-p-convex subset of a topological vector space is

A-p-CS-closed.

There are non-closed A-p-convex sets which are A-p-CS-closed, for example, the open unit ball in a p-normable topological vector space.

Reasoning in a similar way as in [4], we have the following proposition.

PROPOSITION 1. *Let A be a bounded, A-p-convex subset of a topological vector space E. Then, A is A-p-CS-compact if any one of the following conditions is satisfied:*

a) A is A-p-CS-closed and E sequentially complete.

b) A is sequentially complete.

Besides, every A-p-CS-compact set is A-p-CS-closed and bounded.

The following preservation rules are easily verified.

a) If A and B are A-p-CS-compact, so it is $A+B$.

b) The image of an A-p-CS-compact under a continuous linear mapping is A-p-CS-compact.

c) If A and B are A-p-CS-compact, A-p-$co(A\cup B)$ is A-p-CS-compact, too.

d) If A is A-p-CS-compact and B is A-p-CS-closed, $A+B$ and A-p-$co(A\cup B)$ are A-p-CS-closed. An analogous statement holds for a finite number of sets (all but one being A-p-CS-compact).

e) Let E, F be topological vector spaces. If A is an A-p-CS-compact subset of E and S is an A-p-CS-closed subset of $E\times F$, then $S(A)$ is an A-p-CS-closed subset of F.

4. Two theorems about interior points

From the fact that CS-closed sets are semiclosed, Jameson finds interesting results about normed spaces. The following theorem leads us to similar ones for locally bounded topological vector spaces.

THEOREM 1. *Let E be a metrizable topological vector space and A an A-p-CS-closed subset of E. Then, there exists $\alpha\in R$ such that $0<\alpha<1$ and $\alpha I(\overline{A})\subset A$.*

PROOF. Let x be an element of $I(\overline{A})$ and U_n, $n=1, 2, \cdots$ a decreasing base of neighbourhoods of zero. Take

$$V_n=U_n\cap(2^{-\frac{n}{p}}\overline{A}-2^{-\frac{n+1}{p}}x),\ n=1,\ 2,\ \cdots$$

As $2^{-\frac{n}{p}}\overline{A}-2^{-\frac{n+1}{p}}x$ is a neighbourhood of zero for every n and $2^{-\frac{n}{p}}\overline{A}-2^{-\frac{n+1}{p}}x$ contains $2^{-\frac{n+1}{p}}\overline{A}-2^{-\frac{n+2}{p}}x$ for every n, V_n, $n=1, 2, \cdots$, is a decreasing base of neighbourhoods of zero.

Since $0\in 2^{-\frac{1}{p}}\bar{A}-2^{-\frac{2}{p}}x$, there exists an element $a_1\in A$ such that $2^{-\frac{1}{p}}a_1-2^{-\frac{2}{p}}x\in -V_2$. Write $y_1=2^{-\frac{1}{p}}a_1-2^{-\frac{2}{p}}x$.

Since $-y_1\in V_2$, there exists $a_2\in A$ such that $y_1+2^{-\frac{2}{p}}a_2-2^{-\frac{3}{p}}x\in -V_3$. Write $y_2=2^{-\frac{1}{p}}a_1+2^{-\frac{2}{p}}a_2-2^{-\frac{2}{p}}x-2^{-\frac{3}{p}}x$. Then take $y_{n-1}\in -V_n$; there exists $a_n\in A$ such that $y_{n-1}+2^{-\frac{n}{p}}a_n-2^{-\frac{n+1}{p}}x\in -V_{n+1}$. Writing $y_n=\sum_{i=1}^{n}2^{-\frac{1}{p}}a_i-\sum_{i=1}^{n}2^{-\frac{i+1}{p}}x$, we have constructed a sequence that converges to zero. Then, $\sum_1^\infty 2^{-\frac{n}{p}}a_n-\sum_1^\infty 2^{-\frac{n+1}{p}}x=0$, which implies that the series $\sum_1^\infty 2^{-\frac{n}{p}}a_n$ is convergent and being A an A-p-CS-closed set, we have that $\sum_1^\infty 2^{-\frac{n}{p}}a_n=a\in A$. Writing $\alpha=\sum_1^\infty 2^{-\frac{n+1}{p}}$, we have that $\alpha x\in A$.

As a straightforward consequence of the previous theorem, we find:

COROLLARY 1.1. *Let E be a metrizable topological vector space and A a dense, A-p-CS-closed subset of E. Then, $A=E$.*

COROLLARY 1.2. *Let E, F be topological vector spaces, F being metrizable. If A is an A-p-CS-compact subset of E and S an A-p-CS-closed subset of $E\times F$, then there exists $\alpha\in R$ such that $0<\alpha<1$ and $\alpha I(\overline{S(A)})\subset S(A)$.*

COROLLARY 1.3. *Let A, B be subsets of a metrizable topological vector space; A is A-p-CS-compact and B is A-p-CS-closed. Then, there exists $\alpha\in R$ such that $0<\alpha<1$ and $\alpha I(\overline{A+B})\subset A+B$ and $\alpha I\ (\overline{A\text{-}p\text{-}co(AB)})\subset A\text{-}p\text{-}co\ (AB)$.*

In corollary 1.2, if S is a closed subset of $E\times F$, it is possible to omit the metrizability condition of F. We shall prove the following theorem.

THEOREM 2. *Let E, F be topological vector spaces. Suppose that A is an A-p-CS-compact subset of E and that S is a closed, absolutely p-convex subset of $E\times F$. Then, there exists $\alpha\in R$ such that $0<\alpha<1$ and $\alpha I\ (\overline{S(A)})\subset S(A)$.*

PROOF. Let y be an interior point of $\overline{S(A)}$. Then, the sets $V_n=2^{-\frac{n}{p}}\overline{S(A)}-2^{-\frac{n+1}{p}}y$, $n=1, 2, \cdots$, are neighbourhoods of zero in F.

We can construct a sequence of the following form: $y_n=\sum_{i=1}^{n} 2^{-\frac{i}{p}} b_i-\sum_{i=1}^{n} 2^{-\frac{i+1}{p}} y$, such that $y_n\in -V_{n+1}$ and $b_n\in S(A)$. For every $b_n\in S(A)$, there exists $a_n\in A$ such that $(a_n, b_n)\in S$. Since A is A-p-CS-compact, we have that $\sum_{n=1}^{\infty} 2^{-\frac{n}{p}} a_n = a\in A$. Call $\alpha=\sum_{n=1}^{\infty} 2^{-\frac{n+1}{p}}$. Then, $(a, \alpha y)\in S$ and the theorem follows.

To see that $(a, \alpha y)\in S$, let us take U and V, neighbourhoods of zero in E and F, respectively. As A is A-p-CS-compact, there is an $n\in N$ such that $2^{-\frac{n}{p}} A\subset U$ and

$$a-(2^{-\frac{1}{p}} a_1+2^{-\frac{2}{p}} a_2+\cdots+2^{-\frac{n}{p}} a_n)\in U.$$

Besides, $-y_n\in V_n$, which implies that there is an element $t\in S(A)$ such that $2^{-\frac{n}{p}} t\in -y_n+2^{-\frac{n+1}{p}} y+V$. Therefore, there exists an element $a'\in A$ such that

$(a', t)\in S$ and $y_n-2^{-\frac{n}{p}} t+2^{-\frac{n+1}{p}} y\in V$.

Being S an absolutely p-convex set of $E\times F$, we have that

$$(2^{-\frac{1}{p}} a_1+\cdots+2^{-\frac{n}{p}} a_n+2^{-\frac{n}{p}} a',\ y_n+2^{-\frac{2}{p}} y+\cdots+2^{-\frac{n+1}{p}} y+2^{-\frac{n}{p}} t)\in S.$$

Then, there exists an $n\in N$ such that this element is in $(a+U-U)\times(\alpha y+V+V+V)$. So $(a, \alpha y)\in S$.

As a immediate consequence of theorem 2, it is shown:

COROLLARY 2.1. *Let A, B be A-p-convex subsets of a topological vector space E. Suppose that A is A-p-CS-compact and B is closed. Then, $I(\overline{A+B})\neq\phi$ implies $I(A+B)\neq\phi$.*

5. Applications of theorems 1 and 2 to locally bounded topological vector spaces

Recall that a topological vector space is said to be locally bounded if there is a bounded neighbourhood of zero. One basic result about locally bounded spaces is that they are p-normable for a suitable p with $0<p\leq 1$ [6, p. 161].

The open-mapping and the closed-graph theorems for complete p-normable spaces are found in a natural way from theorem 1. Also, we have the following propositions that are proven reasoning similarly to Jameson [4].

PROPOSITION 2. *Every separable p-normable, complete space E is the image of l^p under a continuous, open linear mapping.*

Note. Recall that $l^p=\{(x_n)\in K : \sum_1^\infty |x_n|^p<\infty\}, p\in R,$ such that $0<p\leq 1$ is a complete p-normable space with the p-norm $|||x|||=\sum_1^\infty |x_n|^p$.

PROPOSITION 3. *Let A, B be A-p-CS-closed wedges in a p-normale complete space E. Suppose that for every $x\in E$, there exists bounded sequences (a_n) in A and (b_n) in B such that a_n-b_n converges to x. Then, $E=A-B$ and there is a positive number δ such that $(A\cap U)-(B\cap U)\supset \delta U$, where U denotes the unit closed ball of E.*

PROPOSITION 4. *Let A, B be closed linear subspaces of a p-normable complete space E and let U be the closed unit ball in E. Then, $A+B$ is closed if and only if there is a positive number δ such that $(A\cap U)+(B\cap U)\supset(A+B)\cap\delta U$.*

From theorem 2, the following propositions hold.

PROPOSITION 5. *Let E, F be topological vector spaces, F being complete p-normable. Let T be a linear mapping from E to F with closed graph. If there is a bounded subset B of F such that the closure of $T^{-1}(B)$ is a neighbourhood of zero in E, then T is continuous.*

PROPOSITION 6. Let *(E,τ) be a topological vector space such that the closed, absorbent, absolutely p-convex subsets of E form a base of neighbourhoods of zero for the topology τ. Let F be a complete p-normable space and T a linear mapping from E to F with closed graph. Suppose that there is a bounded subset B of F such that $T(E)$ is contained in $\bigcup_{n=1}^\infty nB$. Then, T is continuous.*

6. The A-p-CS-closure

We call the smallest A-p-CS-closed set containing a given set its *A-p-CS-closure.*

We have the following two propositions.

PROPOSITION 7. *The A-p-CS-closure of a wedge or linear subspace is again a wedge or linear subspace.*

PROPOSITION 8. *Let $\{a_n\}$ be a bounded sequence in a complete p-normable space E. Then, the A-p-CS-closure of $\{a_n\}$ is the set of all sums of absolutely p-convex series $\sum_1^\infty \lambda_n a_n$.*

PROOF. Consider the following mapping T from l^p to E, $T(x)=\sum_1^\infty x_n a_n$. T is a continuous linear mapping.

Consider, now, $U=\{x\in l^p : \sum_1^\infty |x_n|^p<1\}$. Since U is A-p-CS-compact, $T(U)$ is A-p-CS-compact.

Note. The previous proposition is satisfied, too, if we take E to be a sequentially complete, locally convex space.

The following theorem is proved similarly to Jameson.

THEOREM 3. *Let E be a metrizable topological vector space. Then, the A-p-CS-closure of a subset A of E is the set of all sums of convergent absolutely p-convex series of its elements.*

Let E be a vector space and τ_1 and τ_2 two topologies in E. The satatements named below are easily shown.

If τ_2 is finer than τ_1, every A-p-τ_1-CS-closed set is A-p-τ_2-CS-closed and every A-p-τ_2-CS-compact is A-p-τ_1-CS-compact.

Suppose that τ_1 and τ_2 are compatible locally convex topologies for a vector space E. Then, the same sets are A-p-CS-compact with respect to τ_1 and τ_2.

PROPOSITION 9. *If (E,τ) is a locally convex space, then the same sets are A-p-CS-compact with respect to τ and the strong topology given in E by the dual pair $\langle E,E'\rangle$.*

PROPOSITION 10. *Let τ_1 and τ_2 be two compatible locally convex topologies for a vector space E. Then, the same bounded sets are A-p-CS-closed with respect to τ_1 and τ_2.*

REMARK. The author does not know if the same sets are A-p-CS-closed with respect to two compatible locally convex topologies in a vecor space E.

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