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ON PRODUCTS OF BOLZANO-WEIERSTRASS SPACES

By James E. Joseph

A topological space is called *Bolzano-Weierstrass* (*BW*) if each countably infinite subset of the space has a limit point (see [2], [4]). A space is *countably compact* if each countable filterbase on the space has a nonempty adherence. It is well known that every countably compact space is *BW* and that in the presence of T_1-but not in general-every *BW* space is countably compact. Novak [6] has given an example of two T_1 countably compact spaces with a product which is not countably compact. Noble [5] used closed projections to investigate when a product of countably compact spaces is countably compact. In this article we provide several results on the product of *BW* spaces by the use of projections which send subsets without limit points (discrete subsets) onto discrete subsets. We establish that the class of functions from a space X to a space Y which map discrete subsets onto discrete subsets (discrete functions) contains the class of closed functions from X to Y as a by-product of a theorem which gives a new characterization of closed functions. We also give an example to show that continuous open discrete injections into compact Hausdorff spaces may fail to be closed. Unless otherwise stated no separation axioms are assumed in this paper. We will denote the closure and set of limit points of a subset K of a space by $\mathrm{cl}[K]$ and $\mathrm{d}[K]$, respectively, and π with an appropriate subscript will represent the projection mapping from a given product.

DEFINITION. A function $g: X \to Y$ is *discrete* (*countably discrete*) if $\mathrm{d}[g(M)] = \phi$ whenever $\mathrm{d}[M] = \phi$ (and M is countable).

Our first theorem will be utilized as a tool to enable us to each our results for products.

THEOREM 1. *If $g: X \to Y$ is countably discrete and $g^{-1}(y)$ is BW for each $y \in Y$, then $g^{-1}(K)$ is BW for each BW subset K of Y with $g^{-1}(K)$ closed.*

PROOF. Let $M \subset g^{-1}(K)$ be countably infinite. If $g(M)$ is finite there is a $y \in K$ with $M \cap g^{-1}(y)$ infinite. Since $g^{-1}(y)$ is *BW* we have $\mathrm{d}[M \cap g^{-1}(y)] \cap g^{-1}(y) \neq \phi$ so $\mathrm{d}[M] \cap g^{-1}(K) \neq \phi$. If $g(M)$ is infinite then $g(M) \subset K$ so $\mathrm{d}[g(M)] \neq \phi$ and, consequently, $\mathrm{d}[M] \neq \phi$. Since $M \subset g^{-1}(K)$ and $g^{-1}(K)$ is closed we have $\phi \neq \mathrm{d}[M] \subset g^{-1}(K)$. The proof is complete.

Ryll-Nardzewski [7] has shown that if X and Y are countably compact and the projection $\pi_y: X \times Y \to Y$ is a closed function, then $X \times Y$ is countably compact and Atsuji [1] has shown that this condition is necessary. We prove the following analogue for *BW* spaces and countably discrete projections as a corollary to theorem 1.

COROLLARY 1. *If $X \times Y$ has at least one point which is closed then $X \times Y$ is BW if and only if X and Y are BW and $\pi_y: X \times Y \to Y$ is countably discrete.*

PROOF. The sufficiency follows from theorem 1 without requiring that some point of $X \times Y$ be closed. As for the necessity if $\{(x, y)\}$ is closed in $X \times Y$ then $\{x\} \times Y$ and $X \times \{y\}$ are closed subsets of $X \times Y$ and since closed subspaces of *BW* spaces are *BW* the proof is complete.

The following example shows that it was essential that some point be closed in $X \times Y$ in corollary 1.

EXAMPLE 1. Let N be the set of positive integers. Let the space X be N with $\{\{2n-1, 2n\} : n \in N\}$ as a base for the topology and let Y be N with the discrete topology. Then $X \times Y$ is *BW* but Y is not *BW*.

It is interesting that for infinite T_1, *BW*, spaces Y the countable discreteness of $\pi_y: X \times Y \to Y$ forces X to be *BW*. We may therefore prove the following result.

THEOREM 2. *Let Y be infinite and T_1 and suppose that X has at least one point which is closed. Then $X \times Y$ is BW if and only if Y is BW and $\pi_y: X \times Y \to Y$ is countably discrete.*

PROOF. We need show only that X is *BW* if π_y is countably discrete. Let $K \subset X$ be countably infinite and suppose that $\mathrm{d}[K] = \phi$. Let $h: K \to Y$ be an injection and let $M = \{(x, h(x)) : x \in K\}$. Then $d[M] = \phi$ which implies that $\mathrm{d}[h(K)] = \mathrm{d}[\pi_y(M)] = \phi$. This contaadicts the hypothesis that Y is *BW*. Hence $\mathrm{d}[K] \neq \phi$ and X is *BW*. The proof is complete.

Theorem 2 has the following result for arbitrary products as a consequence.

COROLLARY 2. *Let $\{X_\alpha\}_\Sigma$ be a family of spaces and suppose some point in $Z = \prod_\Sigma X_\alpha$ is closed. Suppose for some $\mu \in \Sigma$, X_μ is T_1 and infinite. Then Z is BW if and only if X_μ is BW and $\pi_\mu: Z \to X_\mu$ is countably discrete.*

PROOF. Let $X=\prod_{\Sigma-\{\mu\}} X_\alpha$ and let $Y=X_\mu$. Then X and Y satisfy the hypothesis of theorem 2 and hence $X\times Y$ is BW. Since $X\times Y$ and Z are homeomorphic then Z is BW. The proof is complete.

COROLLARY 3. *Let $\{X_\alpha\}_\Sigma$ be a family of T_1 spaces. Suppose for some $\mu\in\Sigma$, X_μ is infinite. Then $\prod_\Sigma X_\alpha$ is countably compact if and only if X_μ is countably compact and $\pi_\mu: \prod_\Sigma X_\alpha\to X_\mu$ is discrete.*

In our next theorem we give a characterization of closed functions from which it will be obvious that all closed functions are discrete.

THEOREM 3. *A function $g: X\to Y$ is closed if and only if* $\mathrm{d}[g(M)]\subset g(\mathrm{d}[M])$ *for every $M\subset X$.*

PROOF. The sufficiency is fairly immediate. For the necessity let $M\subset X$ and suppose $y\in\mathrm{d}[g(M)]$. Then $y\in\mathrm{cl}[g(M)-\{y\}]=\mathrm{cl}[g(M-g^{-1}(y))]$ and $\mathrm{cl}[g(M-g^{-1}(y))]\subset g(\mathrm{cl}[M-g^{-1}(y)])$ since g is closed. Hence $g^{-1}(y)\cap\mathrm{cl}[M-g^{-1}(y)]\neq\phi$. So some $x\in g^{-1}(y)$ satisfies $x\in\mathrm{cl}[M-\{x\}]$, i.e. $x\in\mathrm{d}[M]$. The proof is complete.

COROLLARY 4. *closed functions are discrete.*

The following example shows that discrete functions are not necessarily closed.

EXAMPLE 2. Let X be a countably compact, locally compact Hausdorff space which is not compact (e.g., the set of all ordinals less than the first uncountable ordinal endowed with the order topology); let X^* be the 1-point compactification of X. The identity function from X to X^* is continuous, open, discrete and is not closed.

Employing Theorem 3 we give a theorem for BW spaces which parallels a result due to Frolik [3]. This theorem should be compared with Theorem 1 above.

THEOREM 4. *If $g: X\to Y$ is closed and $g^{-1}(y)$ is BW for each $y\in Y$, then $g^{-1}(K)$ is BW for each BW subset K of Y.*

PROOF. Let $M\subset g^{-1}(K)$ be countably infinite. If $g(M)$ is finite we procede as in the proof of theorem 1. If $g(M)$ is infinite, then $g(M)\subset K$ and so $\mathrm{d}[g(M)]\cap K\neq\phi$. By theorem 3 we have $g(\mathrm{d}[M])\cap K\neq\phi$; so $\mathrm{d}[M]\cap g^{-1}(K)\neq\phi$. This completes the proof.

We gather the following corollaries from corollary 4 or from theorem 4 since in both cases below the projection $\pi_y : X\times Y\to Y$ is closed.

COROLLARY 5. *If X is compact and Y is BW then $X\times Y$ is BW.*

COROLLARY 6. *If X is countably compact and Y is first countable and BW then $X\times Y$ is BW.*

COROLLARY 7. *Let $\{X_\alpha\}_\Sigma$ be a familx of T_1 spaces. Suppose for some $\mu\in\Sigma$, X_μ is infinite. Then $\prod_\Sigma X_\alpha$ is countably compact if X_μ is countably compact and $\pi_\mu : \prod_\Sigma X_\alpha\to X_\mu$ is closed.*

In closing we give some sufficient conditions for a discrete function to be closed. We recall that a function $g : X\to Y$ has a *closed graph* if whenever a net $\{x_n\}$ in X satisfies $x_n\to x$ and $g(x_n)\to y$ we must have $g(x)=y$. We call a space a *Fréchet space* if a point lies in the closure of a set if and only if there is a sequence in the set converging to the point.

THEOREM 5. *If X is T_1 and Y is a Fréchet space then $g : X\to Y$ with a closed graph is closed if and only if g is discrete.*

PROOF. In view of corollary 4 it is only necessary to prove that if g is discrete, then g is closed. Let $A\subset X$ be closed and let $y\in \mathrm{d}[g(A)]$. There is a sequence $\{x_n\}$ in $A-g^{-1}(y)$ with $g(x_n)\to y$. Let $M=\{x_n : n\in N\}$. Then $y\in \mathrm{d}[g(M)]$, so $\mathrm{d}[M]\neq\phi$. Let $x\in \mathrm{d}[M]$; then there is a subnet $\{x_\alpha\}$ of $\{x_n\}$ with $x_\alpha\to x$. $\{g(x_\alpha)\}$ is a subnet of $\{g(x_n)\}$, so $g(x_\alpha)\to y$ and hence $g(x)=y$ since g has a closed graph. Therefore $x\in g(A)$ and the proof is complete.

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Howard Universiiy
Washington, D. C. 20059

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