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T_2, R_1, and Semi-R_1 Spaces

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0. Abstract

In this paper T_0-identification spaces are used to prove that the semi-R_1 separation axiom is not a generalization of the R_1 separation axiom and to determine conditions, which together with R_1, do and do not imply semi-R_1.

1. Introduction

Semi open sets were first introduced and investigated in 1963 [6]. Since 1963 semi open sets have been used to define and investigate many new topological properties. In 1975 semi-T_i, $i=0, 1, 2$, was defined by replacing the word open in the definition of T_i, $i=0, 1, 2$, by semi open, respectively, and it was proven that semi-T_i, $i=0, 1, 2$, is strickly weaker than T_i, $i=0, 1, 2$, respectively [7]. The semi-R_1 separation axiom was defined and investigated in 1978 [4]. In this paper the relationships between R_1 and semi-R_1 are investigated.

Listed below are definitions and theorems that will be utilized in this paper.

DEFINITION 1.1. If (X, T) is a space and $A \subset X$, then A is *semi open*, denoted by $A \in \mathrm{SO}(X, T)$, iff there exists $O \in T$ such that $O \subset A \subset \bar{O}$ [6].

DEFINITION 1.2. Let (X, T) be a space and let $A, B \subset X$. Then A is *semi closed* iff $X-A$ is semi open and the semi closure of B, denoted by scl B, is the intersection of all semi closed sets containing B [1].

DEFINITION 1.3. A space (X, T) is R_1 iff for $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$ there exist disjoint open sets U and V such that $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$ [2].

DEFINITION 1.4. A space (X, T) is *semi-R_1* iff for $x, y \in X$ such that scl $\{x\} \neq$ scl $\{y\}$ there exist disjoint semi open sets U and V such that scl $\{x\} \subset U$ and scl $\{y\} \subset V$ [4].

DEFINITION 1.5. Let (X, T) be a space and let R be the equivalence relation on X defined by xRy iff $\overline{\{x\}} = \overline{\{y\}}$. Then the *$T_0$-identification space* of (X, T)

is (X_0, S_0), where X_0 is the set of equivalence classes of R and S_0 is the decomposition topology on X_0 [8].

Note that the natural map $P:(X, T)\to(X_0, S_0)$ is closed, opened, and $\overset{-1}{P}(P(O))=O$ for all $O\in T$.

DEFINITION 1.6. A space (X, T) is *extremely disconnected* iff for each $O\in T$, $\bar{O}\in T$ [8].

THEOREM 1.1. *A space (X, T) is R_1 iff (X_0, S_0) is T_2* [5].

THEOREM 1.2. *If (X, T) is R_1, then $X_0=\{\overline{\{x\}}\mid x\in X\}$* [3].

THEOREM 1.3. *A space (X, T) is semi-T_2 iff it is semi-R_1 and semi-T_0* [4].

THEOREM 1.4. *Every T_2 space is semi-T_2* [7].

THEOREM 1.5. *If (X, T) is a space and $A\subset X$, then* scl $A\subset\bar{A}$ [1].

Let S_1 be the statement "Every R_1 space is semi-R_1."

2. Equivalent T_2 condition for S_1 and several applications

Let S_2 be the statement "If (X, T) is T_2 and $x\in X$ such that $\{x\}\notin T$, then there exist disjoint open sets U and V such that $x\in\bar{U}\cap\bar{V}$."

THEOREM 2.1. *S_1 iff S_2.*

PROOF. Assume S_1. Let (X, T) be T_2 and let $x\in X$ such that $\{x\}\notin T$. Let $y\notin X$, let $Y=X\cup\{y\}$, and let $S=\{O\in T\mid x\notin O\}\cup\{O\cup\{y\}\mid x\in O\in T\}$. Then S is a topology on Y and (Y_0, S_0) is homeomorphic to (X, T), which implies (Y_0, S_0) is T_2 and (Y, S) is R_1. Since $\{x\}\notin T$, then $\{x, y\}=\overline{\{y\}}_Y\notin S$ and $y\notin(Y-\overline{\{y\}}_Y)\cup\{x\}\in SO(Y, S)$, which implies scl $\{y\}\neq$scl $\{x\}$. Since (Y, S) is R_1, then (Y, S) is semi-R_1 and there exist disjoint semi open sets A and B such that scl $\{x\}\subset A$ and scl $\{y\}\subset B$. Let $U, V\in S$ such that $U\subset A\subset\bar{U}_Y$ and $V\subset B\subset\bar{V}_Y$. Then $x\notin U\cup V$, which implies $U, V\in T$, and since $(X, T)=(X, S_X)$, then $x\in(\bar{U}_Y\cap X)\cap(\bar{V}_Y\cap X)=\bar{U}_X\cap\bar{V}_X$.

Conversely, suppose S_2. Let (X, T) be R_1. Let $x, y\in X$ such that scl $\{x\}\neq$ scl $\{y\}$. If $\overline{\{x\}}\neq\overline{\{y\}}$, then there exist disjoint open sets U and V such that $\overline{\{x\}}\subset U$ and $\overline{\{y\}}\subset V$, which implies scl $\{x\}\subset U$ and scl $\{y\}\subset V$, where U and V are disjoint semi open sets. Thus consider the case that $\overline{\{x\}}=\overline{\{y\}}$. Since scl $\{x\}\neq$scl $\{y\}$, then $\overline{\{x\}}\notin T$. Since (X, T) is R_1, then (X_0, S_0) is T_2. Let $C_x\in X_0$ such that

$x \in C_x$. Then $C_x = \overline{\{x\}}$ and since $\overline{\{x\}} \notin T$, then $\{C_x\} \notin S_0$. Thus there exist disjoint open sets $\mathscr{U}$ and $\mathscr{V}$ in X_0 snch that $C_x \in \overline{\mathscr{U}} \cap \overline{\mathscr{V}}$. Then $P^{-1}(\mathscr{U})$ and $P^{-1}(\mathscr{V})$ are disjoint open sets in X, $x \in P^{-1}(\overline{\mathscr{U}}) = \overline{P^{-1}(\mathscr{U})}$, and $y \in P^{-1}(\overline{\mathscr{V}}) = \overline{P^{-1}(\mathscr{V})}$, which implies $P^{-1}(\mathscr{U}) \cup \{x\}$ and $P^{-1}(\mathscr{V}) \cup \{y\}$ are disjoint semi open sets. If $z \in \overline{\{y\}} - \{y\}$, then $P^{-1}(\mathscr{U}) \cup \{z\}$ is semi open and does not contain y, which implies $z \notin \mathrm{scl}\ \{y\}$, and since $\mathrm{scl}\ \{y\} \subset \overline{\{y\}}$, then $\mathrm{scl}\ \{y\} = \{y\}$. Similarly $\mathrm{scl}\ \{x\} = \{x\}$ and $\mathrm{scl}\ \{x\} \subset P^{-1}(\mathscr{U}) \cup \{x\}$ and $\mathrm{scl}\ \{y\} \subset P^{-1}(\mathscr{V}) \cup \{y\}$.

Theorem 2.1 can be combined with the following example to prove S_1 is false.

EXAMPLE 2.1. Let N denote the set of natural numbers with the usual topology and let βN denote the Stone-Čech compactification of N. Then βN is extremely disconnected and contains nonisolated points [8]. Thus for each non isolated point $x \in \beta N$, there does not exist disjoint open sets U and V such that $x \in \overline{U} \cap \overline{V}$.

Example 4.1 in [7] can be used to show that semi-R_1 does not imply R_1. Thus R_1 and semi-R_1 are independent.

In [4], it was shown that the T_0-identification space of a semi-R_1 space is semi-T_2. The fact that S_1 is false can be combined with theorem 1.1 and theorem 1.4 to show that the converse of the above statement is false.

3. $\mathscr{P}_0$, $\mathscr{P}_1$, and $\mathscr{P}_2$ sets

Let $\mathscr{P}_1 = \{P \mid P$ is a topological property such that every R_1 space with property P is semi-$R_1\}$ and let $\mathscr{P}_2 = \{P \mid P$ is a topological property such that if (X, T) is T_2 and has property P, then for each $x \in X$ such that $\{x\} \notin T$ there exist disjoint open sets U and V such that $x \in \overline{U} \cap \overline{V}\}$. Since $T_2 \in \mathscr{P}_1$ and $T_2 \notin \mathscr{P}_2$, then $\mathscr{P}_1 \neq \phi$ and $\mathscr{P}_1 \neq \mathscr{P}_2$.

Let DNB be the property "Every point has a decreasing neighborhood base."

THEOREM 3.1. *DNB*$\in \mathscr{P}_2$.

PROOF. Let (X, T) be a T_2 space with the DNB property and let $x \in X$ such that $\{x\} \notin T$. Net $\mathscr{N}$ be a decreasing nbh base of x, let $\geq$ be a well ordering of $\mathscr{N}$, and let F be the first element of $\mathscr{N}$. For each $N \in \mathscr{N}$, let $\mathscr{S}_N = \{O \in \mathscr{N} \mid O \leq N\}$. Then for each $N \in \mathscr{N}$ there exists $f_N : \mathscr{S}_N \to \mathscr{N} \times T \times T$ such that (1) $f_N(F) = f_F(F) = (F, U_F, V_F)$, where U_F and V_F are disjoint open subsets of F and $x \notin \overline{U}_F \cup \overline{V}_F$, and if $F < O \leq N$, then $f_N(O) = f_O(O) = (\hat{O}, U_O, V_O)$, where $\hat{O}$ is the least element of $\{W \in \mathscr{N} \mid W \subset O$ and $W \cap [(\bigcup_{R<O} U_R) \bigcup_{R<O} V_R)] = \phi\}$ and U_O and V_O are disjoint open subsets of $\hat{O}$ such that $x \notin \overline{U}_O \cup \overline{V}_O$, if $x \notin \overline{\bigcup_{R<O} U_P} \cap \overline{\bigcup_{R<O} V_R}$, and $\hat{O} = F$, $U_O = U_F$, and $V_O = V_F$ otherwise, and (2) $(\bigcup_{O \leq N} U_O) \cap (\bigcup_{O \leq N} V_O) = \phi$. The proof is by transfinite induction.

Since (X, T) is T_2 and $\{x\} \notin T$, then every nbh of x contains infinfinitely many points, which implies there exist disjoint open sets U_F, $V_F \subset F$ such that $x \notin \overline{U}_F \cup \overline{V}_F$. Then $f_F : \mathscr{S}_F \rightarrow \mathscr{N} \times T \times T$ defined by $f_F(F) = (F, U_F, V_F)$ satisfies the desired properties.

Assume the statement is true for all $W \in \mathscr{N}$ less than N. If $x \in \overline{\bigcup_{R<N} U_R} \cap \overline{\bigcup_{R<N} V_R}$, then $f_N : \mathscr{S}_N \rightarrow \mathscr{N} \times T \times T$ defined by

$$f_N(O) = \begin{cases} f_O(O) & \text{if } O < N \\ f_F(F) & \text{if } O = N \end{cases} \quad \text{satisfies the desired properties.}$$

Thus consider the case that $x \notin \overline{\bigcup_{R<N} U_R} \cup \overline{\bigcup_{R<N} V_R}$. Then $x \notin \overline{\bigcup_{R<N} U_R}$, for suppose not. Let $O \in T$ such that $x \in O$. Then there exists $W \in \mathscr{N}$ snch that $W \subset O$. Let P be the least element of $\{B < N \mid W \cap (\bigcup_{R \leq B} U_R) \neq \phi\}$ and let S be the immediate successor of P. Then $S < N$. Since $x \notin \overline{\bigcup_{R<S} U_R} \cap \overline{\bigcup_{R<S} V_R}$, then $f_S(S) = (\hat{S}, U_S, V_S)$, where $\hat{S}$ is the least element of $\{Y \in \mathscr{N} \mid Y \subset S$ and $Y \cap [(\bigcup_{R<S} U_R) \cup (\bigcup_{R<S} V_R)] = \phi$ and U_S, V_S are disjoint open subsets of $\hat{S}$ such that $x \notin \overline{U}_S \cup \overline{V}_S$. Since $\hat{S} \subset W$ or $W \subset \hat{S}$, $\hat{S} \cap (\bigcup_{R<S} U_R) = \phi$, and $W \cap (\bigcup_{R<S} U_R) = \phi$, then $\hat{S} \subset W$ and $V_S \subset \hat{S} \subset W \subset O$. Hence $x \in \overline{\bigcup_{R<N} V_R}$, which is a contradiction. By a similar argument $x \notin \overline{\bigcup_{R<N} V_R}$. Let $\hat{N}$ be the least element of $\{Y \in \mathscr{N} \mid Y \subset N$ and $Y \cap [(\bigcup_{R<N} U_R) \cup (\bigcup_{R<N} V_R)] = \phi\}$ and let U_N, V_N be disjoint open subsets of $\hat{N}$ such that $x \notin \overline{U}_N \cup \overline{V}_N$. hTen $f_N : \mathscr{S}_N \rightarrow \mathscr{N} \times T \times T$ defined by

$$f_N(O) = \begin{cases} f_O(O) & \text{if } O < N \\ (\hat{N}, U_N, V_N) & \text{if } O = N \end{cases} \quad \text{satisfies the desired properties.}$$

Thus by transfinite induction the statement is true for each $N \in \mathscr{N}$. Then $A = \bigcup_{N \in \mathscr{N}} U_N$ and $B = \bigcup_{N \in \mathscr{N}} V_N$ are disjoint open sets such that $x \in \overline{A} \cap \overline{B}$.

Let $\mathscr{P}_0 = \{P \mid P$ is a topological property and (X, T) has propperty P iff (X_0, S_0) has property $P\}$

THEOREM 3.2. *Compactness, separability, extremely disconnectedness, and DNB are elements of $\mathscr{P}_0$.*

The straightforward proof is omitted.

THEOREM 3.3. $\mathscr{P}_1 \cap \mathscr{P}_0 = \mathscr{P}_2 \cap \mathscr{P}_0$.

The proof is similar to that for theorem 2.1 and is omitted.

Combining theorem 3.3, theorem 3.2, theorem 3.1, and example 2.1 proves DNB$\in \mathscr{P}_1$ and compactness, separability, and extremely disconnectedness are not elements of $\mathscr{P}_1$.

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