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WALLMAN'S TYPF ORDER COMPACTIFICATIONS II [1)]

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0. Introduction.

Nielsen and Sloyer [7] has introduced the concepts of ideals of semi-continuous functions, and showed the family of all maximal ideals in the semi-ring of all nonnegative lower semi-continuous functions on a T_1-space X is a compactification of X under the Stone topology. Also, Brümmer [1] proved that this compactification of X is equivalent to Wallman compactification of X. In [3], as a generalization of Wallman compactification, the ordered compactification $w_0(X)$ of an ordered topological space $(X,\mathscr{T},\leq)$ with a semi-closed order is constructed.

In this paper, by using the concepts of bi-ideals ([2]), we construct an ordered compactification $\mathfrak{M}_0(X)$ of an ordered topological space $(X,\mathscr{T},\leq)$ with a semi-closed order. Moreover, we show that two ordered compactifications $\mathfrak{M}_0(X)$ and $w_0(X)$ are order equivalent (i.e. they are iseomorphic). We note that these results reduce to the previous mentioned ones([1], [7]) in the case of a discrete order.

Let $(X,\leq)$ be a partiallty ordered set and A a subset of X. Then we write $d(A)=\{y\in X : y\leq x$ for some $x\in A\}$, $i(A)=\{y\in X : x\leq y$ for some $x\in A\}$.

In particular, if A is a singleton, say $\{x\}$, then we write $d(x)$ (resp. $i(x)$). A subset A of X is said to be *decreasing*(resp. *increasing*) if $A=d(A)$ (resp. $A=i(A)$). The order is called *discrete* if $x\leq y$ only when $x=y$. A map f from a partially ordered set X to a partially ordered set Y is said to be *increasing* (resp. *decreasing*) if $x\leq y$ in X implies $f(x)\leq f(y)$ (resp. $f(x)\geq f(y)$) in Y. By an ordered topological space we mean a set X endowed with both a topology $\mathscr{T}$ and a partial order$\leq$. For such an ordered topological space $(X,\mathscr{T},\leq)$, let $\mathscr{U}=\{U\in\mathscr{T} : U$ is increasing$\}$,

$\mathscr{L}=\{U\in\mathscr{T} : U$ is decreasing$\}$.

Then $\mathscr{U}$ and $\mathscr{L}$ are evidently topologies for X, which are called the *upper*, *lower* topologies respectively ([2], [8]). We say that an ordered topological

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space X is *convex* if X has a subbase consisting of the sets in $\mathscr{L}$ and $\mathscr{U}$, or equivalently, if every open set in X can be written as the intersection of an open increasing set and an open decreasing set. The order is said to be *upper* (resp. *lower*) *semi-closed* if, given any $x \in X$, $i(x)$ (resp. $d(x)$) is closed. The order is *semi-closed* if it is both upper and lower semi-closed ([6]).

1. Ordered compactifications of lower semi continuous functions.

We recall that a function f from a topological space X into R is lower semi-continuous if and only if for each $r \in R$, $f^{-1}[(r,\infty)]$ is open in X, where R denotes the set of real numbers equipped with the usual topology and order. Let $(X, \mathscr{T}, \leq)$ be an ordered topological space and let

$\mathscr{L}(X) = \{f : f$ is a lower semi-continuous function on $X\}$,

$\mathscr{L}^+(X) = \{f \in \mathscr{L}(X) : f$ is non-negative$\}$,

$\mathscr{L}^i(X) = \{f \in \mathscr{L}^+(X) : f$ is increasing$\}$,

$\mathscr{L}^d(X) = \{f \in \mathscr{L}^+(X) : f$ is decreasing$\}$.

Then $\mathscr{L}^+(X)$, $\mathscr{L}^i(X)$ and $\mathscr{L}^d(X)$ form semi-rings under the usual pointwise operations.

REMARKS 1.1. (1) U is an open increasing (resp. decreasing) set in an ordered topological space X if and only if its characteristic function χ_U belongs to $\mathscr{L}^i(X)$ (resp. $\mathscr{L}^d(X)$).

(2) The idempotent set of $\mathscr{L}^i(X)$ is equal to the family $\{\chi_U : U \in \mathscr{U}\}$ of all characteristic functions of open increasing sets, and dually.

The following definition is due to Nielsen and Sloyer [7].

DEFINITION 1.2. A proper subset I of $\mathscr{L}^i(X)$ is called an *ideal* if it satisfies the following three conditions:

(1) If f and $g \in I$, then $f+g \in I$

(2) If $f \in I$ and $g \in \mathscr{L}^i(X)$, then $gf \in I$

(3) If $f \in I$, then there exists an idempotent $g, g \neq 1$ in $\mathscr{L}^i(X)$ such that $gf = f$, where 1 is defined by $1(x)=1$ for all $x \in X$. Ideals in $\mathscr{L}^d(X)$ are defined analogously.

Let I be an ideal in $\mathscr{L}^i(X)$ and let $f \in \mathscr{L}^i(X) - I$. Then the ideal generated by $I \cup \{f\}$, denoted by (I,f) is clearly the ideal $\{m+lf : m \in I,\ l \in \mathscr{L}^i(X)$ and $g(m+lf) = m+lf$ for some idempotent $g(\neq 1)$ in $\mathscr{L}^i(X)\}$.

REMARKS 1.3. (1) Let $(X, \mathscr{T})$ be a T_1-space. Then each point of X can be associated with a maximal ideal in $L^+(X)$ (i.e. the set of all non-negative lower semi-continuous functions on X) ([7]). But, this statement need not be true in an ordered topological space $(X, \mathscr{T}, \leq)$ with a semi-closed order. For example, let $X=[0,1]$ be the unit interval equipped with the usual topology and order. For each $x \in X$, let $I_x=\{f \in \mathscr{L}^i(X) : f(x)=0\}$. Then I_x is an ideal of $\mathscr{L}^i(X)$, but not maximal for each $x \in [0,1]$. We also note that if the order on $(X, \mathscr{T}, \leq)$ is discrete, then $L^+(X)=\mathscr{L}^i(X)$.

(2) The set $\{Z(f) : f \in \mathscr{L}^i(X)\}$ is precisely the set of all closed decreasing sets in X, and the set $\{Z(f) : f \in \mathscr{L}^d(X)\}$ is exactly the set of all closed increasing sets in X, where $Z(f)=\{x \in X : f(x)=0\}$.

(3) If f and g are elements of $\mathscr{L}^+(X)$, then $Z(f+g)=Z(f) \cap Z(g)$ and $Z(fg)=Z(f) \cup Z(g)$.

The following definition is due to Canfell [2].

DEFINITION 1.4. Let I and J be ideals in $\mathscr{L}^i(X)$ and $\mathscr{L}^d(X)$ respectively. The pair (I, J) is said to be a *bi-ideal* in $(\mathscr{L}^i(X), \mathscr{L}^d(X))$ if given $i \in I$ and $j \in J$, $Z(i) \cap Z(j) \neq \phi$. For given two bi-ideals (I_1, J_1) and (I_2, J_2), we define a relation $(I_1, J_1) \subseteq (I_2, J_2)$ if and only if $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$. By maximal bi-ideal in $(\mathscr{L}^i(X), \mathscr{L}^d(X))$ we mean a bi-ideal not contained in any other bi-ideal under the above relation.

REMARK 1.5. We note, by Zorn's lemma, that every bi-ideal is contained in a maximal bi-ideal.

Let $\mathfrak{M}_0(X)$ denote the set of all maximal bi-ideals in $(\mathscr{L}^i(X), \mathscr{L}^d(X))$.

The proofs of following two lemmas are similar to those in [2], we will omit them.

LEMMA 1.6. *Let $(M, N) \in \mathfrak{M}_0(X)$ and let $f \in \mathscr{L}^i(X)$. Then $f \in M$ if and only if for given $m \in M$ and $n \in N$, $Z(f) \cap Z(m) \cap Z(n) \neq \phi$, and dually for N.*

LEMMA 1.7. *Let $(M, N) \in \mathfrak{M}_0(X)$. Then the following statements hold.* (1) *Let f and f' be elements of $\mathscr{L}^i(X)$ and $ff' \in M$. Then either $f \in M$ or $f' \in M$, and dually for N.*

(2) *Let $f \in \mathscr{L}^i(X)$ and $g \in \mathscr{L}^d(X)$. Then $fg=0$ implies either $f \in M$ or $g \in N$.*

Define $f^d=\{(M, N) \in \mathfrak{M}_0(X) : f \in M\}$ for given $f \in \mathscr{L}^i(X)$, and
$g^i=\{(M, N) \in \mathfrak{M}_0(X) : g \in N\}$ for given $g \in \mathscr{L}^d(X)$.

Then $\{f^d : f \in \mathscr{L}^i(X)\}$ forms a base for the closed sets in $\mathfrak{M}_0(X)$, since $f^d \cup f'^d=(ff')^d$. Similarly, $\{g^i : g \in \mathscr{L}^d(X)\}$ also forms a base for the closed sets in $\mathfrak{M}_0(X)$. We denote the topologies in $\mathfrak{M}_0(X)$ which have $\{f^d : f\in\mathscr{L}^i(X)\}$ and $\{g^i : g\in\mathscr{L}^d(X)\}$ as basis respectively by $\mathfrak{M}_{\mathscr{L}}$ and $\mathfrak{M}_{\mathscr{U}}$. Let $\mathfrak{M}$ be the smallest topology containing $\mathfrak{M}_{\mathscr{L}}$ and $\mathfrak{M}_{\mathscr{U}}$. Define a relation $\leq$ on $\mathfrak{M}_0(X)$ as follows: $(M,N)\leq(M',N')$ if and only if $M\supseteq M'$ and $N\subseteq N'$ for each (M,N) and (M',N') in $\mathfrak{M}_0(X)$. Then $\leq$ is obviously a partial order on $\mathfrak{M}_0(X)$ and $(\mathfrak{M}_0(X),\mathfrak{M},\leq)$ is an ordered topological space. Also it is immediate that f^d and g^i are closed decreasing and increasing sets in $(\mathfrak{M}_0(X),\mathfrak{M},\leq)$, respectively, for given $f\in\mathscr{L}^i(X)$ and $g\in\mathscr{L}^d(X)$. Hence, we note that $\mathfrak{M}_0(X)$ is a convex ordered topological space.

LEMMA 1.8. *The convex ordered topological space* $(\mathfrak{M}_0(X),\mathfrak{M},\leq)$ *is* T_1*-compact.*

PROOF. It is similar to those in [2].

The following definition is due to Canfell [2].

DEFINITION 1.9. Let (I,J) be a bi-ideal in $(\mathscr{L}^i(X), \mathscr{L}^d(X))$. Then the bi-ideal (I,J) is said to be *fixed* if there exists a point $p\in X$ such that $p\in\bigcap\{Z(i),Z(j) : i\in I \text{ and } j\in J\}$

LEMMA 1.10. *Let* $(X,\mathscr{T},\leq)$ *be an ordered topological space with a semi-closed order. For each* $p\in X$, *define* $M_p^i=\{f\in\mathscr{L}^i(X) : f(p)=0\}$ *and* $M_p^d=\{g\in\mathscr{L}^d(X) : g(p)=0\}$. *Then* (M_p^i,M_p^d) *is a fixed bi-ideal with a point* p.

PROOF. This is immediate from the definition.

PROPOSITION 1.11. *Let* $(X,\mathscr{T},\leq)$ *be an ordered topological space with a semi-closed order. Then the fixed maximal bi-ideals in* $(\mathscr{L}^i(X), \mathscr{L}^d(X))$ *are precisely the pairs* (M_p^i, M_p^d) *for* $p\in X$. *Moreover, these bi-ideals are distinct for distinct points in* X.

PROOF. Let (M,N) be a fixed maximal bi-ideal with a point p in X. Then it is easy to see that $(M,N)\subseteq(M_p^i,M_p^d)$. By Lemma 1.10, (M_p^i,M_p^d) is a fixed bi-ideal with a point p. To show maximality of (M_p^i,M_p^d), let $f\in\mathscr{L}^i(X)$ and $f\notin M_p^i$; then $f(p)>0$. Since $i(p)$ is a closed increasing set, $X-i(p)$ is an open decreasing set. By Remarks 1.1, $\chi_{X-i(p)}\in\mathscr{L}^d(X)$. Suppose that $Z(f)\cap Z(\chi_{X-i(p)})\neq\phi$, say, $x\in Z(f)\cap Z(\chi_{X-i(p)})$. It follows that $x\notin X-i(p)$, and hence $p\leq x$. hence $f(p)=0$, which is a contradiction. Therefore, we have $Z(f)\cap Z(\chi_{X-i(p)})=\phi$.

We also note that $\chi_{X-i(p)}$ belongs to M_p^d. Hence $((M_p^i, f),M_p^d)$ is not a bi-ideal. We can easily observe that a dual result holds for $g\in\mathscr{L}^d(X)$ and $g\notin M_p^d$. Therefore (M_p^i, M_p^d) is a maximal bi-ideal. Hence the fixed maximal bi-ideals in $(\mathscr{L}^i(X), \mathscr{L}^d(X))$ are precisely the pairs (M_p^i, M_p^d) for $p\in X$. Let $p\neq q$ in X. Then we may assume without of generality that $q\not\leq p$. Hence $p\notin i(q)$. By Remarks 1.1, $\chi_{X-i(q)}\in\mathscr{L}^d(X)$, and hence $\chi_{X-i(q)}(q)=0$ and $\chi_{X-i(q)}(p)\neq 0$. Hence $\chi_{X-i(q)}\in M_q^d$, but $\chi_{X-i(q)}\notin M_p^d$. It follows that $(M_p^i, M_p^d)\neq(M_q^i, M_q^d)$. This completes the proof.

PROPOSITION 1.12. *Let $(X, \mathscr{T}, \leq)$ be a compact ordered space with a semi-closed order. Then every bi-ideal in $(\mathscr{L}^i(X), \mathscr{L}^d(X))$ is fixed.*

PROOF. Let (I, J) be a bi-ideal in $(\mathscr{L}^i(X), \mathscr{L}^d(X))$. Then the family $\{Z(i), Z(j) : i\in I, j\in J\}$ has the finite intersection property. By compactness of X, the proposition immediately follows.

REMARK 1.13. From Propositions 1.11 and 1.12, we note that if an ordered topological space $(X, \mathscr{T}, \leq)$ is compact with a semi-closed order, then every maximal bi-ideal is of the form (M_p^i, M_p^d) for $p\in X$.

THEOREM 1.14. *Let $(X, \mathscr{T}, \leq)$ be a convex ordered topological space with a semi-closed order. Then $(X, \mathscr{T}, \leq)$ is iseomorphic to a dense subspace of $(\mathfrak{M}_0(X), \mathfrak{M}, \leq)$.*

PROOF. Define a map $e : (X, \mathscr{T}, \leq)\longrightarrow(\mathfrak{M}_0(X), \mathfrak{M}, \leq)$ by $e(p)=(M_p^i, M_p^d)$ for each $p\in X$, Then, by Lemma 1.8, e is clearly injective. Let $p\leq q$ in X. Then $M_p^i\supseteq M_q^i$ and $M_p^d\subseteq M_q^d$. Hence $(M_p^i, M_p^d)\leq(M_q^i, M_q^d)$, that is, $e(p)\leq e(q)$. Hence e is an increasing function. Let $e(p)\leq e(q)$ in $\mathfrak{M}_0(X)$, and assume that $p\not\leq q$. Then $p\notin d(q)$. Hence $\chi_{X-d(q)}\in\mathscr{L}^i(X)$, $\chi_{X-d(q)}(p)=1$ and $\chi_{X-d(q)}(q)=0$. It follows that $\chi_{X-d(q)}\in M_q^i$ and $\chi_{X-d(q)}\notin M_p^i$. Hence $M_q^i\subseteq M_p^i$, which is a contradiction; therefore $p\leq q$. Hence e is an order isomorphism. To show that X is dense in $\mathfrak{M}_0(X)$, it suffices to prove that if $f\in\mathscr{L}^i(X)$ and $g\in\mathscr{L}^d(X)$, then $\overline{[Z(f)]}=f^d$ in $(\mathfrak{M}_0(X), \mathfrak{M}_{\mathscr{L}})$ and $\overline{[Z(g)]}=g^i$ in $(\mathfrak{M}_0(X), \mathfrak{M}_{\mathscr{U}})$, where $-$ denotes closure in the given spaces respectively, because if $f=g=0$ then $\overline{X}=\mathfrak{M}_0(X)$. Since $Z(f)\subseteq f^d$ and f^d is $\mathfrak{M}_{\mathscr{L}}$-closed, $\overline{[Z(f)]}\subseteq f^d$. On the other hand, suppose that $f'^d\supseteq Z(f)$ for some $f'\in\mathscr{L}^i(X)$. Then $Z(f')=X\cap f'^d\supseteq Z(f)$. Let $(M, N)\in f^d$. Then $f\in M$. Since $Z(f')\supseteq Z(f)$, by Lemma 1.6, we have $f'\in M$; hence $(M, N)\in f'^d$. It follows that $f'^d\supseteq f^d$. Therefore $\overline{[Z(f)]}=f^d$ in $(\mathfrak{M}_0(X), \mathfrak{M}_{\mathscr{L}})$. Finally, it is not difficult to show $\mathfrak{M}|X=\mathscr{T}$. This completes the proof.

COROLLARY 1.15. *Let $(X, \mathscr{T}, \leq)$ be a convex ordered topological space with a semi-closed order. If $(X, \mathscr{T})$ is compact, then $(X, \mathscr{T}, \leq)$ is iseomorphic to $(\mathfrak{M}_0(X), \mathfrak{M}, \leq)$.*

PROOF. It is immediate from Theorem 1.14 and Remark 1.13.

REMARK 1.16. If the given order on X in Theorem 1.14 is discrete, then the theorem reduces to the main result of Nielsen and Sloyer [7].

2. Equivalence of the two ordered compactifications $\omega_0(X)$ and $\mathfrak{M}_0(X)$

In [3], we constructed the ordered compactification $\omega_0(X)$ for a convex ordered topological space $(X, \mathscr{T}, \leq)$ with a semi-closed order. In this section, we investigate relationship between $\omega_0(X)$ and $\mathfrak{M}_0(X)$. In fact, it turns out that they are order equivalent. Throughout this section, we use the same notations and terminologies as those given in [3].

Let $(X, \mathscr{T}, \leq)$ be an ordered topological space with a semi-closed order. Let I be an ideal in $\mathscr{L}^i(X)$ and $\mathscr{F}$ a closed filter in $(X, \mathscr{U})$. We denote $Z(I)=\{Z(f): f \in I\}$ and $Z^{-1}(\mathscr{F})=\{f \in \mathscr{L}^i(X): Z(f) \in \mathscr{F}\}$.

LEMMA 2.1. *Let $(X, \mathscr{F}, \leq)$ be an ordered topological space with a semi-closed order. For any ideal I in $\mathscr{L}^i(X)$, let $\mathscr{F}$ be the filter generated by $Z(I)$, in symbols, $\mathscr{F}=\mathscr{S}([Z(I)])$. Then $\mathscr{F}$ is a closed filter in $(X, \mathscr{U})$.*

PROOF. We show that $Z(I)$ is a filter base for $\mathscr{F}$, consisting only of decreasing closed sets. Obviously, $Z(f)$ is a decreasing closed set for each $f \in I$. Let $Z(f) \in Z(I)$. Then it is easy to see that $Z(f) \neq \phi$. If $Z(f)$ and $Z(g)$ belong to $Z(I)$, then by Remark 1.3, $Z(f) \cap Z(g) \in Z(I)$. Hence $Z(I)$ is a filter base for $\mathscr{F}$, that is, $\mathscr{F}$ is a closed filter in$(X, \mathscr{U})$.

LEMMA 2.2. *Let $(X, \mathscr{T}, \leq)$ be an ordered topological space with a semi-closed order, and let $\mathscr{F}$ be a closed filter in $(X, \mathscr{U})$. Then $Z^{-1}(\mathscr{F})$ is an ideal in $\mathscr{L}^i(X)$. Moreover $\mathscr{F}=\mathscr{S}([Z(Z^{-1}(\mathscr{F}))])$.*

PROOF. It is easy.

REMARK 2.3. We note that Lemmas 2.1 and 2.2 hold dually for $\mathscr{L}^d(X)$ and $(X, \mathscr{L})$.

By the lemmas and remark, we have the following two lemmas:

LEMMA 2.4. *Let $(X, \mathscr{T}, \leq)$ be an ordered topological space with a semi-closed order, and let (M, N) a maximal bi-ideal in $(\mathscr{L}^i(X), \mathscr{L}^d(X))$. If $\mathscr{F}$ and $\mathfrak{G}$*

are the filters generated by the families $\{Z(f): f \in M\}$ *and* $\{Z(g): g \in N\}$ *respectively, that is,* $\mathscr{F} = \mathscr{S}(Z(M))$ and $\mathfrak{G} = \mathscr{S}(Z(N))$, *then* $(\mathscr{F}, \mathfrak{G})$ *is a maximal bi-filter in* X.

LEMMA 2.5. *Let* $(X, \mathscr{T}, \leq)$ *be an ordered topological space with a semi-closed order. Let* $(\mathscr{F}, \mathfrak{G})$ *be a maximal bi-filter in* X. *Then* $(Z^{-1}(\mathscr{F}),\ Z^{-1}(\mathfrak{G}))$ *is a maximal bi-ideal in* $(\mathscr{L}^i(X),\ \mathscr{L}^d(X))$.

THEOREM 2.6. *Let* $(X, \mathscr{T}, \leq)$ *be a convex ordered topological space with a semi-closed order. Then there exists an iseomorphism* $\tilde{\Phi}$: $\mathfrak{M}_0(X) \longrightarrow \omega_0(X)$ *such that the commutativity relation* $\tilde{\Phi} \circ e = \Phi$ *holds in the following triangle*:

$$\begin{array}{ccc} X & \xrightarrow{\ e\ } & \mathfrak{M}_0(X) \\ & \Phi \searrow \quad \swarrow \tilde{\Phi} & \\ & \omega_0(X) & \end{array}$$

PROOF. Define $\Phi: \mathfrak{M}_0(X) \longrightarrow \omega_0(X)$ by $\tilde{\Phi}((M, N)) = (\mathscr{S}[Z(M)],\ \mathscr{S}[Z(N)])$ for any $(M, N) \in \mathfrak{M}_0(X)$. Then this is well-defined by Lemma 2.4. Now, we shall show that $\tilde{\Phi}$ is a required iseomorphism. Firstly, we show that $\tilde{\Phi} \circ e = \Phi$: Let $x \in X$, and let $A \in \mathscr{S}[d(x)]$. Then $d(x) \subseteq A$. Since $d(x) = Z(\chi_{X-d(X)})$, we have $\chi_{X-d(x)} \in \mathscr{L}^i(X)$. Hence $\chi_{X-d(x)} \in M^i_x$. Thus we have $A \in \mathscr{S}[Z(M^i_x)]$. Conversely, let $A \in \mathscr{S}[Z(M^i_x)]$. Then $A \supseteq Z(f)$ for some $f \in M^i_x$, and hence $x \in Z(f)$. Since $Z(f)$ is a closed decreasing set, $d(x) \subseteq Z(f) \subseteq A$. Hence $A \in \mathscr{S}[d(x)]$. Thus we have showed that $\mathscr{S}[d(x)] = \mathscr{S}[Z(M^i_x)]$ for each $x \in X$. Similarly, we can show that $\mathscr{S}[i(x)] = \mathscr{S}[Z(M^d_x)]$. Thus we have $(\mathscr{S}[d(x)],\ \mathscr{S}[i(x)]) = (\mathscr{S}[Z(M^i_x)],\ \mathscr{S}[Z(M^d_x)])$ for each $x \in X$. Hence $(\tilde{\Phi} \circ e)(x) = \tilde{\Phi}[e(x)] = \tilde{\Phi}[(M^i_x, M^d_x)] = (\mathscr{S}[Z(M^i_x)],\ \mathscr{S}[Z(M^d_x)]) = (\mathscr{S}[d(x)],\ \mathscr{S}[i(x)]) = \Phi(x)$. Thus $\tilde{\Phi} \circ e = \Phi$.

Secondly, we show that $\tilde{\Phi}$ is an order isomorphism: Let (M_1, N_1) and (M_2, N_2) be in $\mathfrak{M}_0(X)$, and let $\tilde{\Phi}[(M_1, N_1)] = \tilde{\Phi}[(M_2, N_2)]$, that is, $(\mathscr{S}[Z(M_1)],\ \mathscr{S}[Z(N_1)] = (\mathscr{S}[Z(M_2)], \mathscr{S}[Z(N_2)]$. We can easily see that $M_1 \subseteq Z^{-1}(\mathscr{S}[Z(M_1)]$ and $N_1 \subseteq Z^{-1}(\mathscr{S}[Z(N_1)])$. Since (M_1, N_1) is a maximal bi-ideal, $(M_1, N_1) = (Z^{-1}(\mathscr{S}[Z(M_1)]),\ Z^{-1}(\mathscr{S}[Z(N_1)]))$. Similarly, we have $(M_2, N_2) = (Z^{-1}(\mathscr{S}[Z(M_2)]),\ Z^{-1}(\mathscr{S}[Z(N_2)]))$. It follows that $(M_1, N_1) = (M_2, N_2)$. Hence $\tilde{\Phi}$ is one to one. Let $(\mathscr{F}, \mathfrak{G}) \in \omega_0(X)$. Then by Lemma 2.5, $(Z^{-1}(\mathscr{F}), Z^{-1}(\mathfrak{G})) \in \mathfrak{M}_0(X)$. Hence $\tilde{\Phi}\ [(Z^{-1}(\mathscr{F}),\ Z^{-1}(\mathfrak{G}))] = (\mathscr{S}[Z(Z^{-1}(\mathscr{F}))],\ \mathscr{S}[Z(Z^{-1}(\mathfrak{G}))] = (\mathscr{F}, \mathfrak{G})$ by Lemma 2.2. Thus $\tilde{\Phi}$ is onto. Clearly $\tilde{\Phi}$ is increasing. Let (M_1, N_1) and (M_2, N_2) be in $\mathfrak{M}_0(X)$ and let $\tilde{\Phi}(M_1, N_1) \leq \tilde{\Phi}(M_2, N_2)$ in $\omega_0(X)$. Then it is easy to show that $(M_1, N_1) \leq (M_2, N_2)$. Therefore $\tilde{\Phi}$ is an order isomorphism.

Finally, we show that $\tilde{\Phi}$ is a homeomorphism: For given $f\in\mathscr{L}^i(X)$, let $(M,N)\in f^d$; then $Z(f)\in\mathscr{S}[Z(M)]$. Hence $\tilde{\Phi}[(M,N)]\in Z(f)^d$. Thus we have $\tilde{\Phi}(f^d)\subseteq Z(f)^d$. Conversely, $(\mathscr{F},\mathfrak{S})\in Z(f)^d$. Then $Z(f)\in\mathscr{F}$ or $f\in Z^{-1}(\mathscr{F})$. Hence $(Z^{-1}(\mathscr{F}), Z^{-1}(\mathfrak{S}))\in f^d$, and therefore $\tilde{\Phi}[Z^{-1}(\mathscr{F}), Z^{-1}(\mathfrak{S})]=(\mathscr{F}, \mathfrak{S})\in\tilde{\Phi}(f^d)$. Thus $Z(f)^d\subseteq\tilde{\Phi}(f^d)$. Hence we have $\tilde{\Phi}(f^d)=Z(f)^d$ for given $f\in\mathscr{L}^i(X)$. Dually, we have $\tilde{\Phi}(g^i)=Z(g)^i$ for given $g\in\mathscr{L}^d(X)$. Since $\mathfrak{M}_0(X)$ and $\omega_0(X)$ are convex ordered topological spaces, $\tilde{\Phi}$ is clearly a homeomorphism. Hence $\tilde{\Phi}$ is an iseomorphism from $\mathfrak{M}_0(X)$ onto $\omega_0(X)$. This completes the proof.

REMARK 2.7. If the given order on X in Theorm 2.6 is discrete, then this reduces to the main result of Brümmer [1], that is, $\mathfrak{M}_0(X)$ is the Wallman compactification of a T_1-space X.

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