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ON AN INTEGRAL EQUATION ASSOCIATED WITH A PRODUCTION PROBLEM

By S. L. Kalla, L. Viloria and S. Conde

0. Abstract

We consider the problem that how must the production of certain item vary as a function of time, if for known losses due to depreciation the total amount of the product is to have a constant value. The integral equation associated with the problem is solved by an appeal to the convolution quotients. The production function comes out to be an expression containing the generalized Laguerre polynomials. The loss function and the production function are tabulated for different values of the parameter by using an IBM 370/145 computer.

1. Introduction

Here we consider the problem that how must the production of certain item vary as a function of time, if for known losses due to depreciation the total amount of the product is to have a constant value. Let initially ($t=0$) the total amount of the unused product be M ; the production is to be arranged so that this amount is mantained. Let the loss function be denoted by $f(t)$; it is defined for $t \geqslant 0$ and has the following property : $Mf(t)$ denote the amount lost due to depreciation, if no production takes place for $t>0$.

Over a long time all the amount is used up, that is

$$\int_0^\infty M\, f(t)\; dt=M \tag{1}$$

$$\int_0^\infty f(t)dt=1 \tag{2}$$

The production fuction $g(t)$ denotes the production per unit time at the time t. Thus the production in the time interval $(x, x+\Delta x)$ equals $g(x)\ \Delta x$. Hence at a later instant t the loss is $g(x)\ f(t-x)\ \Delta x$. Hence

$$\int_0^t f(t-x)\ g(x)\ dx \tag{3}$$

is the loss which is to be produced until the instant t. Thus the difference of production and loss

$$\int_0^t g(x)\ dx - \int_0^t f(t-x)\ g(x)\ dx \tag{4}$$

must be equal to the loss $M\ f(t)$. Hence,

$$\int_0^t g(x)\ dx - \int_0^t f(t-x)\ g(x)\ dx = M\ f(t) \tag{5}$$

The solution of the problem is obtained by an appeal to the Mikusinski's operators (convolution quotients) [2, 7]. In a typical example the production function comes out to be an expression containing generalized Laguerre polynomial. The loss function and the production function are tabulated for different values of the parameter by using an IBM 370/145 computer.

2. Convolution quotients

Here we give a brief introduction to the theory of generalized functions due to Mikusinski [7]. Let us consider a set C of locally integrable functions [2]. The operators of addition and convolution product for any two elements $f, g \in C$, are defined as:

$$(f+g)\ (t) = f(t) + g(t) \tag{6}$$

$$f * g = \int_0^t f(x)\ g(t-x) dx \tag{7}$$

The convolution quotients f/g are introduced very much in the same way as rational numbers are introduced as quotients of integers. The set of these elements is called *the convolution field* F. Some elements of F correspond to numbers, others to continuous or discontinuous functions, operators of differentiation and integration also belong to this field. Convolution quotients can be regarded either as generalized functions or as operators.

It is easy to verify that all elements of the set C belong to the field F, however, there are elements in F which may not belong to C. For example, the element $\delta(t) \in F$ satisfies

$$f(t) * \delta(t) = f(t) \tag{8}$$

In fact the property (8) corresponds to the Dirac's δ-function [2]. This $\delta(t)$ is the convolution quotient f/f for every $f \in C$.

By definition, $h(t)$ is the Heaviside's unit function,

$$h(t)=1, \quad t \geqslant 0$$

$$h^2(t)=h(t)*h(t)=t$$

and by induction, we have

$$h^n=\left\{\frac{t^{n-1}}{(n-1)!}\right\} \tag{9}$$

Thus h can be considered as an operator of integration, h^n corresponds to n times repeated integration.

Since for every $f \in C$, $f/f=\delta$, we have

$$h/h=\delta \text{ or } h*h^{-1}=\delta \tag{10}$$

which becomes

$$h*s=\delta, \quad s=h^{-1} \tag{11}$$

The relation (11) is taken as the definition of a new operator. From here it can be inferred that s corresponds to the operator of differentiation. For any element $f \in C$, having a locally integrals derivative of order n [2, 7],

$$s^n f=f^{(n)}+s^0 f^{(n-1)}(0)+\cdots+s^{n-1}f(0) \tag{12}$$

where $f^{(n)}$ are ordinary derivatives. This is called *the extended derivative of order n* of the element f.

In dealing with differential and integral equations in the domain of convolution quotients, an important role is played by the operator s, [2, 4, 5].

We call convolution quotients (distributions or generalized functions) the elements of the field F. In this field all equations

$$f*\xi=g \quad (f,\ g \in C) \tag{13}$$

are solvable and the unique solution is represented symbolically by

$$\xi=g/f \tag{14}$$

3. The solution of the problem

The equation (5) can be written as

$$g*h-f*g=Mf$$

$$g=\frac{Mf}{h-f}=Mf*\frac{\delta}{h-f} \tag{15}$$

$$=Mf*\frac{s}{\delta-sf}$$

$$=Msf*\frac{\delta}{\delta-sf}$$

$$=Mf*\frac{\delta}{\delta-F}$$

where $sf=F$. Hence

$g=MF(\delta+R)$, where R is the resolvent kernel of F [3]

$=M(F+FR)$

$=MR$

$$=MF_1(t)+F_2(t)+\cdots \tag{16}$$

$$\text{where } F_1(t)=F,\ F_r=F*F_{r-1},\ r=2,\ 3,\ \cdots \tag{17}$$

We set

$$f(t)=\frac{\alpha^{\lambda}}{\Gamma(\lambda)}e^{-\alpha t}t^{\lambda-1}, \tag{18}$$

where α and λ are constants.

Clearly this choice of $f(t)$ satisfies the condition (2).

Representing the function $f(t)$ in terms of the operator s[1]

$$f(t)\longleftrightarrow\frac{\alpha^{\lambda}}{(s+\alpha)^{\lambda}} \tag{19}$$

and therefore

$$F(t)\longleftrightarrow\frac{s\alpha^{\lambda}}{(s+\alpha)^{\lambda}} \tag{20}$$

Its r-th iterated kernel is

$$\frac{s^r\alpha^{r\lambda}}{(s+\alpha)^{r\lambda}}\longleftrightarrow\frac{\alpha^{r\lambda}r!}{\Gamma(r\lambda)}t^{r\lambda-r-1}e^{-\alpha t}L_r^{r\lambda-r-1}(-\alpha t) \tag{21}$$

Thus

$$g(t)=M\sum_{r=1}^{\infty}\frac{\alpha^{r\lambda}r!}{\Gamma(r\lambda)}t^{r\lambda-r-1}e^{-\alpha t}L_r^{r\lambda-r-1}(-\alpha t) \tag{22}$$

where $L_r^n(x)$ are the generalized Laguerre polynomials [6].

4. Computation

In the following tables we have tabulated $Mf(t)$ and the production function $g(t)$ for different values of α (0.01 to 0.20, $\Delta\alpha=.01$) at different intervals of time t (from 1 to 60, $\Delta t=1$). The values of M and λ are taken as 10^5 and 2 respectively. In the numerical calculations we have considered so many terms as to give us correct figures upto eight decimal points. Using the Fortran IV-H extended, the calculations were affected at a terminal station COPE 1200 of the IBM 370/145 computer.

Coordinación de Postgrado
Facultad de Ingeniería
La Universidad del Zulia
Maracaibo-Venezuela

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