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THE ORDER OF CYCLICITY OF BIPARTITE TOURNAMENTS AND (0, 1) MATRICES

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0. Abstract

A (0, 1) matrix is acyclic if it does not have a permutation matrix of order 2 as a submatrix. A bipartite tournament is acyclic if and only if its adjacency matrix is acyclic. The concepts of (maximal) order of cyclicity of a matrix and a bipartite tournament are introduced and studied.

1. Introduction

All matrices of this paper are (0, 1). All graphs are directed. We use the graph-theory notation of [4] and the matrix-theory notation of [7].

The *order of cyclicity*, $\mu(G)$, of a graph G, is the smallest number of arcs which must be reversed in G in order to obtain an acyclic graph.

An $(m, n)-$(*bipartite*) *tournament* is an oriented graph G such that $V(G)=R\cup C$, $R=\{r_1, \cdots, r_m\}$, $C=\{c_1, \cdots, c_n\}$ and G has mn arcs, each incident with a point in R and a point in C. In other words, an (m, n)-tournament is obtained by orientation of the complete bipartite graph $K_{m,n}$.

Cyclicity in ordinary tournaments was studied by Bermond [1], Erdös and Moon [3], Jung [5], Kotzig [6] and Spencer [8]. Here we study the order of cyclicity of (m, n)-tournaments.

There is a natural one to one correspondence between $m\times n$ matrices and (m, n) tournaments. The rows of a matrix A correspond to the points $r_1, \cdots, r_m$ and the columns to the points $c_1, \cdots, c_n$ of an (m, n)-tournament G_A, where r_ic_j is an arc of G_A if and only if $a_{ij}=1$. Thus the 2^{mn} matrices of order $m\times n$ describe all possible orientations of $K_{m,n}$. Notice that the adjacency matrix of G_A is

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

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where $b_{ij}=1-a_{ji}$.

A matrix is *acyclic* if it does not have a permutation matrix of order 2 as a submatrix. A *transformation* of a matrix A is replacing it by another matrix which differs from A in exactly one entry. The *order of cyclicity* of A, $\mu(A)$, is the length of the shortest sequence of transformations needed to convert A into an acyclic matrix.

It is easy to see that every (m,n)-trounament which is not acyclic contains a cycle of length four. Thus, G_A is acyclic if and only if A is acyclic and for every A, $\mu(A)=\mu(G_A)$.

For every two natural numbers, m and n, we define

$$\mu(m, n)=\max \mu(A)$$

where the maximum is taken over all $m\times n$ matrices. Clearly, $\mu(m, n)$ is an upper bound for the order of cyclicity of all subgraphs of (m,n)-tournaments.

In this paper we study the function $\mu(m, n)$ for various values of m and n. An upper bound is given is Section 3 following some simple observations. Some exact values of the function are given in Section 4. Two conjectures conclude the paper.

2. Simple observations

In the sequel we shall use the following relations:

(1) $$\mu(A)=\mu(A^T)=\mu(J-A)=\mu(JA)=\mu(PAQ)$$

where J denotes a matrix of ones and P and Q are permutation matrices of the appropriate orders.

(2) $$\mu(A\ B)\geq\mu(A)+\mu(B)$$

(3) $$\mu(m,n)=\mu(n,m)$$

(4) $$\mu(m,l+n)\geq\mu(m,l)+\mu(m,n).$$

The distance, $d(A,B)$ between two $m\times n$ matrices A and B is the minimal number of transformations needed to convert A into B. Thus

(5) $$\mu(A)=\min_{B \text{ is acyclic}} d(A,B).$$

Let f_i denote the m-vector having ones in the first i rows and zeros elsewhere. In particular, f_0 is the zero vector. The *fundamental distance* of an m-vector c, $\phi(c)$, is defined by

(6) $$\phi(c)=\min_{0\leq i\leq m} d(c,f_i).$$

A matrix is in a *fundamental form if*

$$j_2 > j_1 \Longrightarrow A^{j_1} \geq A^{j_2}$$

and

$$i_2 > i_1 \Longrightarrow A_{i_1} \geq A_{i_2}.$$

Etringer and Jackson [2] have shown that a matrix is acyclic if and only if it can be brought by permuting rows and columns into a fundamental form. This, (5) and (6), yield the following formulas.

THEOREM 1. *Let $\mathscr{A}$ denote the set of $m \times n$ matrices and $\mathscr{P}$ the set of permutation matrices of order m. Then*

$$\mu(A) = \min_{P \in \mathscr{P}} \sum_{j=1}^{n} \phi((PA)^j) \tag{7}$$

and

$$\mu(m, n) = \max_{A \in \mathscr{A}} \min_{P \in \mathscr{P}} \sum_{j=1}^{n} \phi((PA)^j).$$

3. An upper bound

Let X_k^m denote the family of $m \times \binom{m}{k}$ matrices which $\binom{m}{k}$ different columns, each having column sum k. Clearly, all matrices in X_k^m have the same order of cyclicity. We denote this number by $\binom{m}{k} \xi_k^m$ and compute ξ_k^m.

It is enough to compute ξ_k^m for $k \leq \left[\frac{m}{2}\right]$, where $[x]$ denotes the largest integer not greater than x, since $\xi_k^m = \xi_{m-k}^m$ by (1). Let $\delta_k^m(i)$ denote the number of columns in any matrix in X_k^m having fundamental distance equal to i. Since X_k^m is invariant under permutations and $\delta_k^m(i) = 0$ for $i > k$, it follows that

$$\binom{m}{k} \xi_k^m = \sum_{i=1}^{k} i \delta_k^m(i). \tag{8}$$

Obviously

$$\delta_k^m(0) = 1.$$

To calculate $\delta_k^m(i)$ for $1 \leq i \leq k$, consider a matrix in X_k^m of the form

$$\begin{pmatrix} A & B \\ 00 \cdots 0 & 11 \cdots 1 \end{pmatrix}.$$

Clearly, $A \in X_k^{m-1}$ and $B \in X_{k-1}^{m-1}$ and by induction

$$\delta_k^m(i) = \binom{m}{i} - \binom{m}{i-1} \quad \text{if} \quad i \leq k. \tag{9}$$

Substituting (9) in (8) we obtain

$$\binom{m}{k}\xi_k^m=\sum_{i=1}^{k}\left[i\binom{m}{i}-i\binom{m}{i-1}\right]=k\binom{m}{k}-\sum_{j=0}^{k-1}\binom{m}{j}.$$

Thus,

$$\xi_k^m=k-\binom{m}{k}^{-1}\sum_{j=0}^{k-1}\binom{m}{j}. \tag{10}$$

This formula implies, via straight computation, the monotonicity of ξ_k^m, namely,

$$k<\left[\frac{m}{2}\right]\Longrightarrow\xi_{k-1}^m<\xi_k^m.$$

Thus max $\xi_k^m=\xi_{\left[\frac{m}{2}\right]}^m$. We denote this value by $\xi(m)$ and compute it from (12).

Consider two cases:

(i) $m=2r$

$$\sum_{j=0}^{r-1}\binom{m}{j}=\frac{1}{2}\left(2^m-\binom{m}{r}\right)$$

(ii) $m=2r+1$

$$\sum_{j=0}^{r-1}\binom{m}{j}=\frac{1}{2}\left(2^m-2\binom{m}{r}\right)$$

In both cases

$$\xi(m)=\frac{m+1}{2}-\frac{2^{m-1}}{l_m} \tag{11}$$

where l_m is the largest binomial coefficient in $(1+1)^m$.

We observe in passing that

$$\xi(2r)-\xi(2r-1)=\frac{1}{2}=\lim_{m\to\infty}\frac{\xi(m)}{m}$$

Several values of $\xi(m)$ are given in the following tableau:

:	2	3	4	5	6	7	8
$\xi(m)$:	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{7}{6}$	$\frac{14}{10}$	$\frac{38}{20}$	$\frac{76}{35}$	$\frac{187}{70}$

to state the result referred to in the tittle of the section.

THEOREM 2.

$$\mu(m,n)\leq\min\{n\xi(m),\ m\xi(n)\} \tag{12}$$

where ξ_m is given in (11). *Equality in* (12) *holds if and only if* $n\equiv0 \bmod l_m$ *or* $m\equiv0 \bmod l_n$.

PROOF. By (3) it is enough to show that

$$\mu(m, n) \leq n\xi(m).$$

Let A be an $m \times n$ matrix with column sums $s_1, \cdots, s_n$. Then, by (7)

$$\mu(A) = \min_{P \in \mathscr{P}} \sum_{j=1}^{n} \phi((PA)^j)$$

(13) $$\leq \frac{1}{m!} \sum_{P \in \mathscr{P}} \sum_{j=1}^{n} \phi((PA)^j)$$

$$= \sum_{j=1}^{n} \sum_{P \in \mathscr{P}} \frac{1}{m!} \phi((PA)^i)$$

(14) $$= \sum_{j=1}^{n} \xi_{s_j}^{m} \leq \sum_{j=1}^{m} \xi(m)$$

Equality in (14) exists if and only if for every j, $s_j \in \left\{\left[\frac{m}{2}\right], -\left[-\frac{m}{2}\right]\right\}$.

Equality in (13) holds if and only if $\sum_{j=1}^{n} \phi((PA)^i)$ does not depend on $\mathscr{P}$.

This completes the proof and the section.

4. Same exact values of $\mu(m, n)$

THEOREM 3.

(15) $$\mu(1, n) = 0$$

$$\mu(2, n) = \left[\frac{n}{2}\right]$$

PROOF. Equation (15) is trivial and is included for the sake of completeness.

By Theorem 2, $\mu(2, n) \leq \left[\frac{n}{2}\right]$.

The example $\begin{pmatrix} 1 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \end{pmatrix}$ demonstrates that $\mu(2, n) \geq \left[\frac{n}{2}\right]$.

COROLLARY.

(16) $$\mu(m, n) \leq \min\{\mu(m-1, n) + \mu(2, n),\ \mu(m, n-1) + \mu(m, 2)\}$$

THEOREM 4.

$$\mu(3, n) = \left[\frac{2}{3} n\right]$$

PROOF. By Theorem 2, $\mu(3, n) \leq \left[\frac{2}{3} n\right]$. The converse follows from

$$\mu\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=2,\ \mu(3,2)=1 \text{ and } (4).$$

THEOREM 5. *Let k be a nonnegative integer.*
Then

$$\mu(4,6k+1)=\mu(3,6k+1)+\mu(2,6k+1)=7k$$
$$\mu(4,6k+2)=\mu(3,6k+2)+\mu(2,6k+2)=7k+2$$
$$\mu(4,6k+3)=\mu(3,6k+3)+\mu(2,6k+3)-1=7k+2$$
$$\mu(4,6k+4)=\mu(3,6k+4)+\mu(2,6k+4)=7k+4$$
$$\mu(4,6k+5)=\mu(3,6k+5)+\mu(2,6k+5)=7k+5$$
$$\mu(4,6k+6)=\mu(3,6k+6)+\mu(2,6k+6)=7k+7$$

PROOF. By (16), $\mu(4,n)\leq\mu(2,n)+\mu(3,n)$, We first show that for $n\neq 3$, mod 6, one has equality.

This is clear for $n=1$ and 2 and 4, since

$$\mu\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}=4$$

For $n=6$ we observe that

$$\mu\begin{pmatrix}1&1&1&0&0&0\\1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{pmatrix}=7$$

by Theorem 2. From this we deduce that $\mu(4,5)=5$, since

$$\mu(4,6)\leq\mu(4,5)+\mu(4,2)\Rightarrow\mu(4,5)\geq 5 \text{ and } \mu(4,5)\leq\mu(2,5)+\mu(3,5)=5.$$

We proceed by induction on k : *Let* $n=6k+b$, $b\neq 3$. Then

$$\mu(4,n+6)\geq\mu(4,n)+\mu(4,6)=\mu(4,n)+7=\mu(3,n)+\mu(2,n)+7.$$

$$\mu(4,n+6)\leq(3,n+6)+\mu(2,n+6)=\mu(3,n)+\mu(2,n)+7. \text{ Equality.}$$

For $n=6k+3$ we first observe that $2=\mu(4,3)=\mu(2,3)+\mu(3,3)-1$. We show that in general. $\mu(4,6k+3)<7k+3$. Suppose $\mu(4,6k+3)=7k+3$. Let A be a $4\times(6k+3)$ matrix with $\mu(A)=7k+3$. The matrix A cannot have two columns with column sum 1 or 3 since in this case

$$\mu(A)\leq 2+\mu(4,6k+1)=7k+2.$$

By an homogenizing a row we mean replacing it by a row of zeros or a row of ones. Suppose now that A has a column with a single zero [one]. One can homogenize the row containing this zero [one], by at most $3k+1$ transformations. Thus, $\mu(A) \leq 3k+1+\mu(3, 6k+2)=7k+2$. Thus all column sums of A are 2. Similarly all row sums are $3k+1$ or $3k+2$. By homogenizing a row, one is left with, say, $3k+1$ columns with sum$=1$ ***and*** $3k+2$ columns with sum$=2$. An easy calculation shows that $\mu(A)<7k+3$ also in this case.

Thus $\mu(4, 6k+3) \leq 7k+2$, but since $\mu(4, 6k+2)=7k+2$. the inequality is an equality.

THEOREM 6.

(17) $$\mu(5, 5)=6$$

(18) $$\mu(5, 6)=7$$

(19) $$\mu(6, 6)=10$$

(20) $$\mu(5, 10)=14$$

(21) $$\mu(5, 9)=12$$

PROOF. First (17). Let

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since all rows and columns play the same role in A,

$$\mu(A)=2+\mu\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}=6.$$

We proceed to (18).

$$\mu(4, 6)=7 \Longrightarrow \mu(5, 6) \geq 7.$$
$$\mu(5, 5)=6 \Longrightarrow \mu(5, 6) \leq 8.$$

The proof that $\mu(5, 6) \neq 8$ involves a large number of case distinctions and will be omitted.

Formula (19) follows from

$$10=\mu(2, 6)+\mu(5, 6) \geq \mu(6, 6) \geq \mu+\mu(4, 6)=10$$

Equation (20) follows from Theorem 2. The same theorem implies that $\mu(5, 9) \leq 12$. But $\mu(5, 9)=11$ contradicts (20). This proves (21) and completes the proof of the theorem.

5. Two conjectures.

The following conjecture is true for $n \leq 4$.

CONJECTURE 1.

$$\mu(m, n+l_m) = \mu(m, n) + l_m \, \xi(m).$$

The paper is concluded with the following suggestion for $n=5$.

CONJECTURE 2.

$\mu(5, 10k) = 14k$ $\quad \mu(5, 10k+1) = 14k$ $\quad \mu(5, 10k+2) = 14k+2$

$\mu(5, 10k+3) = 14k+3$ $\quad \mu(5, 10k+4) = 14k+5$ $\quad \mu(5, 10k+5) = 14k+6$

$\mu(5, 10k+6) = 14k+7$ $\quad \mu(5, 10k+7) = 14k+9$ $\quad \mu(5, 10k+8) = 14k+10$

$\mu(5, 10k+9) = 14k+12.$

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