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IN INTEGRAL TRANSFORM INVOLVING TWO GENERALISED H-FUNCTIONS

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0. Abstract

In the present paper we study a new integral transform whose kernel involves the product of two H-functions of two complex variables. Next, we establish an inversion formula for this new transform. On account of very general nature of its kernel, several other integral transforms studies earlier by many research workers viz., Bose (1952), Mukherji (1962), Nigam (1963), Rathie (1965), Singh (1969), Mittal & Goel (1973), and Gupta, Garg & Kalla (1975), follow as its particular cases.

1. Introduction

Recently Mittal and Goyal [8] have defined the so called *'generalized H-function transform'* involving two variables as follows:

$$\phi_1(p,\ q)=pq\int_0^\infty\int_0^\infty H^*(px,\ qy)\ f(x,\ y)\ dx\ dy \tag{1.1}$$

provided the integral (1.1) is absolutely convergent and the function $H^*(x, y)$ is a special case of the H-function of two variables (with $n_1=0$) defined and represented by Mittal and Gupta [9, p.117]. For convenience and brevity, however, we shall use the contracted notation introduced by Srivastava and Panda [15, p.226, Eq.(1.5) st.seq.] as follows:

$$H(x,\ y)=H^{0,\,n_1\,:\,(m_2,\,n_2)\,:\,(m_3,\,n_3)}_{p_1,\,q_1\,:\,(p_2,\,q_2)\,:\,(p_3,\,q_3)}\left(\begin{matrix}[(a_{p_1}:\alpha_{p_1},A_{p_1})]:[(c_{p_2},\gamma_{p_2})]\,[(e_{p_3},E_{p_3})]: \\ [(b_{q_1}:\beta_{q_1},B_{q_1})]:[(d_{q_2},\delta_{q_2})]\,[(f_{q_3},F_{q_3})]:\end{matrix}\ x,y\right)$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2}U_1(s,t)U_2(s)U_3(t)x^s y^t\ ds\ dt, \tag{1.2}$$

where

$$U_1(s,t)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)}{\prod_{j=1+n_1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)}$$

$$U_2(s)=\frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j s)}{\prod_{j=1+n_2}^{p_2}\Gamma(c_j-\gamma_j s)\prod_{j=1+m_2}^{q_2}\Gamma(1-d_j+\delta_j s)}$$

$$U_3(t)=\frac{\prod_{j=1}^{m_3}\Gamma(f_j-F_j t)\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j t)}{\prod_{j=1+n_3}^{p_3}\Gamma(e_j-E_j t)\prod_{j=1+m_3}^{q_3}\Gamma(1-f_j+F_j t)}$$

provided that x and y are not equal to zero and an empty product is interpreted as unity. The non-negative integers n_i, p_i, q_i $(i=1,2,3)$ and m_2, m_3 are such that

$$0\leq n_i\leq p_i,\ q_i\geq 0,\ 0\leq m_j\leq q_j\ (i=1,2,3\ ;\ j=2,3)\ ;$$

and the letters $\alpha,\beta,\gamma,\delta$ and A,B,E,F all denote positive quantities.

The contours L_1 and L_2 are suitably defined and the poles of the integrand are assumed to be simple.

The conditions for the function $H(x,y)$ being analytic and for the integral (1.2) to converge are given by Mittal and Gupta [9, p.119, conditions (i)-(vi)]. The corresponding appropriate conditions are also satisfied by $H_1(x,y)$ occuring in this paper.

For convenience, let $[(a_p,\alpha_p)]$ and $[(a_p:\alpha_p,A_p)]$ abbreviate the p-parameter sequences $(a_1,\ \alpha_1)$, ..., $(a_p,\ \alpha_p)$ and $(a_1:\alpha_1,\ A_1)$, ..., $(a_p:\alpha_p,\ A_p)$ respectively, with similar interpretations for $[(c_p,\ \gamma_p)]$, $[(b_q:\beta_q,\ B_q)]$, etc.

By summing up the residues at the simple poles of the integrand of (1.2), the following expansion for $H_1(x,\ y)$, defined below, has been derived [5] in terms of Fox's H-function [4] :

$$H_1(x,y)=H^{0,N_1:(M_2,N_2):(1,N_3)}_{P_1,Q_1:(P_2,Q_2):(P_3,Q_3+1)}\left(\begin{matrix}[(a_{p_1}':\alpha_{p_1}',A_{p_1}')]:[(c_{p_2}',\gamma_{p_2}')]:[(e_{p_3}',E_{p_3}')]:\\ [(b_{Q_1}':\beta_{Q_1}',B_{Q_1}')]:[(d_{Q_2}',\delta_{Q_2}')]:(f_0',F_0'),[(f_{Q_3}',F_{Q_3}')]:\end{matrix}\ x,y\right)$$

$$=\frac{1}{F_0}\sum_{s=0}^{\infty}\frac{(-1)^s(y)^{\rho_s}\prod_{j=1}^{N_3}\Gamma(1-e_j'+E_j'\rho_s)}{s!\prod_{j=N_3+1}^{p_3}\Gamma(e_j'-E_j'\rho_s)\prod_{j=1}^{Q_3}\Gamma(1-f_j'+F_j'\rho_s)}$$

$$\cdot H^{M_2,N_1+N_2}_{P_1+P_2,Q_1+Q_2}\left(\times\left|\begin{matrix}[(c_{p_2}',\ \gamma_{p_2}')],\ [(a_{p_1}'-A_{p_1}'\rho_s,\ \alpha_{p_1}')]\\ [(d_{q_2}',\ \delta_{q_2}')],[(b_{q_1}'+B_{q_1}'\rho_s,\ \beta_{q_1}')]\end{matrix}\right.\right)\qquad(1.3)$$

where $\rho_s=(f_0'+s)/F_0'$, provided that the series on the right side of (1.3) is absolutely convergent.

In this paper we introduce a new integral transform defined by

$$\phi_2(u, v)=\int_0^\infty\int_0^\infty (1+\sigma_1 ux)^{-\lambda_2}(1+\sigma_2 vy)^{-\mu_2} H\left(\frac{a(ux)^{h_1}}{(1+\sigma_1 ux)^{h_2}}, \frac{b(uy)^{k_1}}{(1+\sigma_2 vy)^{k_2}}\right)$$

$$\cdot H_1\left(\frac{m(ux)^{v_1}}{(1+\sigma_1 ux)^{v_2}}, \frac{n(vy)^{w_1}}{(1+\sigma_2 vy)^{w_2}}\right) f(x, y)\, dx\, dy \qquad (1.4)$$

provided the integral (1.4) is absolutsly convergent.

2. An integral required

First we establish the following interesting integral which we shall require in the proof of an inversion formula for the integral transform defined by (1.4):

$$I_1=\int_0^\infty\int_0^\infty \frac{x^{\lambda_1} y^{\mu_1}}{(1+\sigma_1 x)^{\lambda_2}(1+\sigma_2 y)^{\mu_2}} H\left(\frac{ax^{h_1}}{(1+\sigma_1 x)^{h_2}}, \frac{by^{k_1}}{(1+\sigma_2 y)^{k_2}}\right)$$

$$\cdot H_1\left(\frac{mx^{v_1}}{(1+\sigma_1 x)^{v_2}}, \frac{ny^{w_1}}{(1+\sigma_2 y)^{w_2}}\right) dx\, dy$$

$$=\frac{m\sigma_1^{-(1+\lambda_1)}\sigma_2^{-(1+\mu_1)}}{F_0}\sum_{s=0}^{\infty}\sum_{w=1}^{M_2}\sum_{r=0}^{\infty}\frac{(-1)^{r+s}f(\rho_s)}{\delta_w' s!\, r!} h(\rho_r')(\sigma_1)^{-v_1}$$

$$\cdot \rho_r' m^{\rho_r'} H^{0,\, n_q+N_1 : (m_2,\, n+N_2+2) : (m_3,\, n_3+2)}_{p_1+p_1,\, q_1 : (p_2+p_2+2,\, q_2+1) : (p_3+2,\, q_3+1)}\left(\begin{matrix}[(a_{p_1}, \alpha_{p_1}, A_{p_1})] : [(a_{p_1}', \alpha_{p_1}', A_{p_1}')] : \\ [(b_{q_1} : \beta_{q_1}, B_{q_1})] \qquad\qquad : \end{matrix}\right.$$

$$\begin{matrix}\cdot(-\lambda_1-v_1\rho'_r, h_2),\ (2-\lambda_2-\lambda_1-v_2\rho_r', h_2-h_1),\ [(c_{p_2}, \gamma_{p_2})],\ [(c'_{p_2}, \gamma'_{p_2})] : \\ (1-\lambda_2-v_2\rho_r', h_2),\ [(d_{q_2}, \delta_{q_2})] \qquad\qquad : \end{matrix}$$

$$\left.\begin{matrix}(\mu_1-w_1\rho_s, k_1),\ (2-\mu_2-\mu_2+(w_2+w_1)\rho_s, k_2-k_1),\ [(e_{p_3}, E_{p_3})] \\ (1-\mu_2, k_2),\ \qquad\qquad [(f_{q_3}, F_{q_3})]\end{matrix}\ a\sigma_1^{-h_1},\ b\sigma_2^{-k_1}\right) \qquad (2.1)$$

where ρ_s, $f(\rho_s)$, ρ_r' and $h(\rho_r')$ are given by,

$$\rho_s=\frac{f'_0+s}{F_0'},\quad f(\rho_s)=\frac{\prod_{j=1}^{N_3}\Gamma(1-e_j'+E_j'\rho_s)}{\prod_{j=N_3+1}^{P_3}\Gamma(e_j'-E_j'\rho_s)\prod_{j=1}^{Q_3}\Gamma(1-f_j'+F_j'\rho_s)}$$

$$\rho_r'=\frac{d_w+r}{\delta_u'}(w=1, \cdots, M_2),$$

$$h(\rho_r')=\prod_{\substack{j=1\\ j=w}}^{M_2}\Gamma(d_j'-\delta_j'\rho_r')[\prod_{j=M_2+1}^{Q_2}\Gamma(1-d_j'+\delta_j'\rho_r')\prod_{j=1}^{Q_1}\Gamma(1-b_j'-B_j'\rho_s+\beta_j'\rho_r')\cdot$$

$$\prod_{j=1}^{P_2}\Gamma(c_j'-\gamma_j'\rho_r')\prod_{j=1}^{P_1}\Gamma(a_j'-A_j'\rho_s-\alpha_j'\rho_r')]^{-1}$$

provided that

$$\sigma_1>0,\ \sigma_2>0,\ h_2>h_1>0,\ k_2>k_1>0,\ w_2>w_1>0,\ v_2>v_1>0.$$

$$\mathrm{Re}[\lambda_2+h_2(d_j/\delta_j)+v_2(d_i'/\delta_i')]>\mathrm{Re}[\lambda_1+h_1(d_j/\delta_j)+v_1(d_i/\delta_i')+1]>0,$$

$$\mathrm{Re}[\mu_2+k_2(f_t/F_t)+w_2(f'_0/F'_0)]>\mathrm{Re}[\mu_1+k_1(f_t/F_t)+w_1(f_0'/F_0')+1]>0,$$

$$(j=1,\cdots,m_2;\ i=1,\cdots,m_3;\ t=1,\cdots,M_2)$$

and the series occuring on the right side of (2.1) is absolutely convergent.

PROOF. To prove (2.1), we first substitute for the H-function occuring on the left hand side in terms of the double contour integral of Mellin Barne's type from (1.2) and then change the order of (x, y) and (s, t)-integrals (which is justified due to their absolute convergence). To evaluate the inner (x, y)-integral, we express H^* in the series form (1.4) involving Fox's H-function and then Fox's H-function thus involved is also expressed in series form by using the result 2,

$$H_{P,Q}^{m,N}\left(\times\middle|\begin{matrix}[(a_p',\alpha_p')]\\ [(b_Q',\beta_Q')]\end{matrix}\right)=\sum_{w=1}^{M}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\beta_w'}g(\rho_r)x^{\rho_r},$$

where ρ_r and $g(\rho_r)$ are given by

$$\rho_r=\frac{b'_w+r}{\beta'_w}(w=1,\cdots,M),$$

$$\text{and } g(\rho_r)=\frac{\prod_{\substack{j=0\\ j\neq w}}^{M}\Gamma(b_j'-\beta_j'\rho_r)\prod_{j=1}^{N}\Gamma(1-a_j'+\alpha_j'\rho_r)}{\prod_{j=M+1}^{P}\Gamma(a_j'-\alpha_j'\rho_r)\prod_{j=M+1}^{Q}\Gamma(1-b_j'+\beta_j'\rho_r)}$$

Now changing the order of integration and summations (which is justified due to uniform convergence of the series and the absolute convergence of the double integral thus involved), evaluating the inner (x, y)-integral with the help of result [3, p. 232], and then interpreting the resulting. Contour integral by means of (1.2), we arrive at the result (2.1.)

3. Inversion Formula

We now established the following inversion formula for the generalized integral transform defined by (1.4).

If $\phi_2[f(x,y)\,;\,u,v]$ is defined by (1.4), then

$$f(x,y)=\frac{1}{(2\lambda i)^2}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} I_2 x^{\lambda_1}y^{\mu_1}d\lambda_1 d\mu_1 \tag{3.1}$$

where

$$I_2=\int_0^\infty\int_0^\infty u^{\lambda_1}v^{\mu_1}\phi_2[f(x,y)\,;\,u,v]\ du\ dv \tag{3.2}$$

and I_1 is given by (2.1), provided that $m,n,u,v,a,$ and b are all positive quantities; $f(x,y)$ is sectionally continuous; the integral transform (1.4) of $f(x,y)$ exists, the double integral (3.2) is absolutely convergent; and

$$\text{Re }[\lambda_2+h_2(d_j/\delta_j)+v_2(d_i'/\delta_i')]>\text{Re}[\lambda_1+h_1(d_j/\delta_j)+v_1(d_i'/\delta_i')+1]>0$$
$$\text{Re }[\mu_2+k_2(f_t/F_t)+w_2(f_0'/F_0')]>\text{Re}[\mu_1+k_1(f_t/F_t)+w_1(f_0'/F_0')+1]>0$$
$$j=1,\cdots,m_2\,;\ \ i=1,\cdots m_3\,;\ \ t=1,\cdots M_2$$

PROOF: By virtue of (1.4), we have

$$I_2=\int_0^\infty\int_0^\infty u^{\lambda_1}v^{\mu_1}\Big\{\int_0^\infty\int_0^\infty(1+\sigma_1 ux)^{-\lambda_2}(1+\sigma_2 vy)^{-\mu_2}$$
$$\cdot H\left(\frac{a(ux)^{h_1}}{(1+0_1ux)^{h_2}},\ \frac{b(vy)^{k_1}}{(1+0_2uy)^{k_2}}\right)\ H_1\left(\frac{m(ux)^{v_1}}{(1+0_1ux)^{v_2}},\ \left(\frac{n(vy)^{w_1}}{(1+0_2vy)^{w_2}}\right)\right.$$
$$\cdot f(g,y)\ dx\ dy\Big\}\ du\ dv \tag{3.3}$$

Now changing the order of (u,v)-integral and (x,y)-Integral(which is justified due to absolute convergence of these integrals), and replacing ux by p and vy by q in (3.3), we obtain

$$I_2=\int_0^\infty\int_0^\infty x^{-(1+\lambda_1)}y^{-(1+\mu_1)}f(x,y)\ dx\ dy\int_0^\infty\int_0^\infty\frac{p^{\lambda_1}\ q^{\mu_1}}{(1+\sigma_1 p)^{\lambda_2}(1+\sigma_2 q)^{\mu_2}}$$
$$\cdot H\left(\frac{ap^{h_1}}{(1+\sigma_1 p)^{h_2}},\ \frac{bq^{k_1}}{(1+\sigma_2 q)^{k_2}}\right)\ H_1\left(\frac{mp^{v_1}}{(1+\sigma_1 p)^{v_2}},\ \frac{nq^{w_1}}{(1+\sigma_2 q)^{w_2}}\right)dp\ dq. \tag{3.4}$$

On evaluating the (p,q)-integral with the help of (2.1), we get

$$I_2=I_1\int_0^\infty\int_0^\infty x^{-(1+\lambda_1)}y^{-(1+\mu_1)}f(x,y)\ dx\ dy \tag{3.5}$$

Now, using the familiar Reeds theorem I, II [13], the inversion formula (3.1) is completely established under the stated conditions.

4. Particular cases

(i) If we set $n_1=0$, $\sigma_1=\sigma_2=0$, $h_1=k_1=1$, $v_1=w_1=1$, $P_1=Q_1=N_1=0$, $M_2=Q_2=1$, $N_2=P_2=0$, $d_1'=0$, $\delta_1'=1$, $P_3=Q_3=N_3=0$, $f_0'=0$, $F_0'=1$ in (1.4), we obtain the $H^*(x,y)$-transform recently defined by Gupta, Gargand Kalla [6] by virtue of the familier relations:

$$H_{0,1}^{1,0}(px \mid \overline{(0,1)})=e^{-px}.$$

(ii) In addition to above, if we put $a=b=1$, $m=n=0$, in (1.2), it reduces to the double H-function transform defined by (1.1) which further reduces to the integral transforms recently studied by various authors notably Singh [14], Nigam [11], Mukherjee [10], Bose [10], Rathie [12], etc.

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