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ON THE FUNCTIONAL EQUATION $f[x+y\,f(x)]=f(x)\,f(y)$

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1. Introduction

The functional equation $f: R \longrightarrow R$, R the set of real numers, such that

$$f[x+y\,f(x)]=f(x)\,f(y) \qquad (1)$$

was first studied in [2]. The equation has certain applications in the continuous groups and geometric objects. Practically the only continuous solutions of the equation are given in the following theorem.

THEOREM *A* (See [1], page 132-135): *Let* $f: R \longrightarrow R$, f *satisfy* (1). *Then followings are the only continuous solutions of* (1).

i) $f(x)=0,\ \forall\ x\in R$

ii) $f(x)=cx+1,\ \forall x\in R,$ *where* $c\in R$

iii) $f(x)=\begin{cases}1-\dfrac{x}{x_1} & \forall\ x\leqslant x_1\\ 0 & \forall\ x>x_1>0\end{cases}$

iv) $f(x)=\begin{cases}0 & \forall\ x\leqslant x_2<0\\ 1-\dfrac{x}{x_2} & \forall\ x\geqslant x_2\end{cases}$

The proof of the theorem A is based on the following lemma.

LEMMA *B*. (See [1], page 132-135): *Let* f *a continous solution of* (1). *If there exist* $x_1, x_2\in R, x_1\neq x_2$ *such that* $f(x_1)=f(x_2)\neq 0$, *then* $f(x)=constant$.

The Dirichlet function, which takes value 1 for the rational numbers and 0 for the irrational numbers, is an example of bounded, measurable solution of (1). Using Hamel Base, it is posible to construct a unbounded and nonmeasurable solution of (1). In this article we generalize the domain of the equation (1) to a real topological vector space X.

2. We use X to denote a real topological vector space and X^* the dual space of X; that is the collection of all real continouous linear functionals on X. A subset $M\subseteq X$ is called a *linear manifold* if $M=x_0+M_0$, where $x_0\in X$ and M_0 a vector subspace of X. M is called a *hyperplane* if M_0 is maximal. The closure

$\overline{M}$ of M is also a linear manifold if M is; hence any hyperplane is closed or dense in X. For any hyperplane M, there exists ($\phi : X \longrightarrow R$, ϕ linear, such that $M = x_0 + \{x | \phi(x) = 0\} = \{x | \phi(x) = \phi(x_0)\}$ for some $x_0 \in X$, and M is closed iff $\phi \in X^*$ (i.e., ϕ is linear and continuous).

LEMMA 1. *Let* $f : X \longrightarrow R$ *be continuous nontrivial such that* $f[x+y\ f(x)] = f(x) f(y)$. *Then* $M_0 = \{x | f(x) = 1\}$ *is a closed hyperplane containing* 0.

PROOF. If x_0, $y_0 \in M_0$, then $f(x_0+y_0) = f[x_0+y_0\ f(x_0)] = f(x_0)\ f(y_0) = 1$, hence $x_0+y_0 \in M_0$. Putting $x=y=0$, we get $f(0) = f(0)^2$ and this implies $f(0)=0$ or $f(0) = 1$. If $f(0)=0$, then $f(x)=0$ $\forall x \in X$ and we have to omit this case. Hence $f(0) = 1$, or $0 \in M_0$.

Let $x_0 \in M_0$ Then $f(x_0)\ f(-x_0) = f(0) = 1$. Hence $f(-x_0) = 1$ or $-x_0 \in M_0$.

Let $x_0 \in M_0$, $x_0 \neq 0$. Define $g_{x_0} : R \longrightarrow R$, $g_{x_0}(\lambda) = f(\lambda x_0)$ $\forall \lambda \in R$; then g_{x_0} satisfies (1). But $g_{x_0}(0) = g_{x_0}(1) = 1$, hence by Lemma B, we have $g_{x_0}(\lambda) = \text{constant} = 1$, and this implies $f(\lambda x_0) = 1$, *or* $\lambda x_0 \in M_0$ $\forall$ $\lambda \in R$. Hence M_0 is a vector subspace of X. M_0 is closed by the continuity of f. If M_0 is not maximal, then there exist x_1, $x_2 \in X$ linearly independent such that $\text{span}\{x_1, x_2\} \cap M_0 = \{0\}$, here $\text{span}\{x_1, x_2\} = \{x | x = \lambda x_1 + \mu x_2,\ \lambda, \mu \in R\}$. Define $g_1(\lambda) \stackrel{\text{def.}}{=} f(\lambda x_1)$, $g_2(\lambda) \stackrel{\text{def.}}{=} f(\lambda x_2)$ $\forall$ $\lambda \in R$. Then g_1 and g_2 satisfy (1). Hence there exist $z_1 = \lambda_1 x_2$, $z_2 = \lambda_2 x_2$ such that $f(z_1) < 1$, $f(z_2) > 1$, which would imply the existence of $z_0 \in \{z | z = \lambda z_1 + (1-\lambda) z_2,\ 0 \leq \lambda \leq 1\}$ such that $f(z_0) = 1$. Thus we reach to a contradiction.

THEOREM 1. *Let* X *a real topological vector space such that* $f[x+y\ f(x)] = f(x)\ f(y)$. *Then the following are the only continuous solutions.*

1. $f(x) = 0$ $\forall$ $x \in X$.
2. $f(x) = \phi(x) + 1$ $\forall x \in X$. *where* $\phi \in X^*$
3. $f(x) = \begin{cases} 1 - \phi(x), & \forall x \in \{x | \phi(x) \leqslant 1\} \\ 0, & \forall x \in \{x | \phi(x) \geqslant 1\} \end{cases}$ *where* $\phi \in X^*$.

PROOF. Let $M_0 = \{x | f(x) = 1\}$. Then M_0 is a closed, maximal hypperplane containing 0. Let $\phi_0 \in X^*$ such that $M_0 = \{x \in X | \phi_0(x) = 0\}$, Let $x_0 \in X$ such that $\phi_0(x_0) = 1$. Then $X = M \oplus R x_0$, the direct sum, where $R x_0 = \{r x_0 | r \in R\}$. Let X/M_0 the quotient space. We define $\tilde{f} : X/M_0 \longrightarrow R$, $\tilde{f}(\tilde{x}) \stackrel{\text{def.}}{=} f(u)$, $u \in \tilde{x} \in X/M_0$, where $\tilde{x}$ is an equivalence class. $\tilde{f}$ is well defined because $f(m+x) = f(x)$ $\forall$ $m \in M_0$, $\forall x \in X$.

For $\tilde{x}, \tilde{y}\in X/M_0$, supposing $u\in\tilde{x}$, $v\in\tilde{y}$, then $u+v\,f(u)\in\tilde{x}+\tilde{y}\tilde{f}(\tilde{x})$, which implies

$$\tilde{f}[\tilde{x}+\tilde{y}\tilde{f}(\tilde{x})]=f[u+vf(u)]=f(u)\,f(v)=\tilde{f}(\tilde{x})\,\tilde{f}(\tilde{y}).$$

X/M_0 is of dimension 1, and we can assume $X/M_0=\{\lambda\tilde{x}_0 \mid \lambda\in R\}$.

Since $\tilde{f}$ is continuous and satisfies (1), then the solutions of $\tilde{f}$ are the following.

i) $\tilde{f}(\lambda\tilde{x}_0)=0, \ \forall\lambda\in R$

ii) $\tilde{f}(\lambda\tilde{x}_0)=c\lambda+1, \ \forall\ \lambda\in R$, where $c\in R$.

iii) $\tilde{f}(\lambda\tilde{x}_0)=\begin{cases}1-\dfrac{\lambda}{\lambda_1}, & \forall\ \lambda\leqslant\lambda_1\\ 0, & \forall\ \lambda\geqslant\lambda_1>0\end{cases}$

iv) $\tilde{f}(\lambda\tilde{x}_0)=\begin{cases}0, & \forall\ \lambda\leqslant\lambda_2<0\\ 1-\dfrac{\lambda}{\lambda_2}, & \forall\ \lambda\geqslant\lambda_2\end{cases}$

Hence the solutions of f take the forms:

1. $f(x)=0 \quad \forall x\in X.$
2. $f(x)=c\,\phi_0(x)+1=\phi(x)+1 \quad \forall x\in X$ with $\phi=c\,\phi_0\in X^*$

$$\left.\begin{array}{l} 3a.\ f(x)=\begin{cases}1-\dfrac{\phi_0(x)}{\lambda_1} & \forall x\in\{x\mid\phi_0(x)\leqslant\lambda_1\}\\ 0 & \forall x\in\{x\mid\phi_0(x)\geqslant\lambda_1>0\}\end{cases}\\ 3b.\ f(x)=\begin{cases}0 & \forall x\in\{x\mid\phi_0(x)\leqslant\lambda_2<0\}\\ 1-\dfrac{\phi_0(x)}{\lambda_2} & \forall\ x\in\{x\mid\phi_0(x)\geqslant\lambda_2\}\end{cases}\end{array}\right\}\Longleftrightarrow (3.)$$

3. $f(x)=\begin{cases}1-\phi(x) & \forall x\in\{x\mid\phi(x)\leqslant 1\}\\ 0 & \forall x\in\{x\mid\phi(x)\geqslant 1\}\end{cases}$ with $\phi\in X^*$

On the other hand, it is easy to prove that (1), (2), (3) satisfy the functional equation

$$f[x+yf(x)]=f(x)\,f(y)$$

THEOREM 2. *Let* f: $H\longrightarrow R$ *and* f *satisfies* $f[x+y\,f(x)]=f(x)\ f(y)\ \forall x,\ y\in H$, *where* H *is a real Hilbert space. Then the only continuous solutions of* f *are the following*:

1. $f(x)=0\ \forall\ x\in H$
2. $f(x)=\langle x, x_0\rangle+1\ \forall x\in H$ *with* $x_0\in H$

3. $f(x)=\begin{cases}1-\langle x, x_0\rangle & \forall x\in\{x\in H \mid \langle x_0, x\rangle\leqslant 1\} \\ 0 & \forall x\in\{x\in H \mid \langle x, x_0\rangle\geqslant 1\},\ x_0\in H,\end{cases}$

where $\langle,\rangle$ *denotes the inner product of* H.

The proof follows directly from theorem 1 and the Riez Representation Theorem.

COROLLARY. *Let* $f: R^n \longrightarrow R$, *and* f *satisfy* $f[x+yf(x)]=f(x)\ f(y)$, $\forall x, y\in R^n$. *Then the only continuous solutions of* f *are*:

1) $f(x)=0 \quad \forall\ x=(x_1, x_2, \cdots, x_n)\in R^n$.

2) $f(x)=1+\sum_{i=1}^{n} c_i\ x_i \quad \forall x=(x_1, x_2, \cdots, x_n)\in R^n$

3) $f(x)=\begin{cases}1-\sum_{i=1}^{n} c_i x_i & \forall\ x\in\{(x_i, \cdots, x_n) \mid \sum_{i=1}^{n} c_i x_i\leqslant 1\} \\ 0 & \forall\ x\in\{(x_1, x_2, \cdots, x_n) \mid \sum_{i=1}^{n} c_i x_i\geqslant 1\}\end{cases}$

where $c_i\in R$, $i=1, 2, \cdots, n$.

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