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CERTAIN INTEGRAL TRANSFORMS AND HEAT CONDUCTION IN A TRUNCATED WEDGE OF INFINITE HEIGHT

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1. Introduction

The problems concerning conduction of heat in solids in the interior of which heat is generated or absorbed are of considerable importance in technical applications. Thus heat may be produced by the passage of an electric current, radio active decay, chemical reaction, hydration of cement, ripening of apples etc. Nuclear reactors and space research also give rise to different problems of heat transfer. Sneddon [12] considered the conduction of heat in certain types of cylinders. Marchi and Zgrablich [7] solved the problem of finding the temperature at any point of a hollow cylinder of any height with heat radiation on its surfaces by the extended finite Hankel transform and sine transform. Since then, Bajpai [1], Bhonsle [2], Kalla et al. [6], Mathur [8], Mehta [9] and several others have also considered the problems of conduction of heat in solids having sources of heat in their interior.

Recently, Goyal and Vasishta [5] solved a problem of finding the temperature at any point of a truncated wedge of finite height with heat radiation on its surfaces and having sources of heat within it.

The object of this paper is to solve a problem, which enable us to find the temperature at any point of a truncated wedge of infinite height having sources of heat within it. There is heat radiation on its outside and inside surfaces. The problem is solved by the application of the powerful tool of integral transforms. In the problem, first we take the source of heat as an arbitrary function and next, as a special case, we consider a moving source of heat whose position at any instant 't' is given by $z=vt$. The source itself at that instant is expressed as $\delta(z-vt)$, wherein δ stands for the well known Dirac-delta function. Later on, another useful special case of the problem is mentioned and an interesting example to illustrate the same is also given at the end of the paper.

2. Formulation of the problem

We consider the distribution of temperature in a truncated wedge of infinite height define by $a \leq r \leq b$, $0 \leq \theta \leq \varepsilon$, having sources of heat within it. There is heat radiation at the faces $r=a$ and $r=b$ into a medium at constant temperature. The temperature $u \equiv u(r, \theta, z, t)$ in this case, at any point of the wedge (where t is time) is governed by the following partial differential equation [9, p.400]:

$$(2.1) \qquad \frac{\partial u}{\partial t} = \beta\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}\right) + \phi(r, \theta, z, t)$$

where β is a constant, known as diffusivity. The boundary conditions of the problem are

$$(2.2) \qquad u \text{ and } \frac{\partial u}{\partial z} \longrightarrow 0 \text{ as } |z| \to \infty, \quad a<r<b, \quad 0<\theta<\varepsilon, \quad t>0$$

$$(2.3) \qquad k_1 \frac{\partial u}{\partial r} + u = u_1 \,;\; r=a, \quad 0<\theta<\varepsilon, \quad t>0$$

$$(2.4) \qquad k_2 \frac{\partial u}{\partial r} + u = u_2 \,;\; r=b, \quad 0<\theta<\varepsilon, \quad t>0$$

where k_1 and k_2 are radiation constants.

$$(2.5) \qquad u \to 0 \,;\; \theta=0 \text{ (or } \theta=\varepsilon), \quad a<r<b, \quad t>0$$

$$(2.6) \qquad u=E(r, \theta, z) \,;\; a<r<b, \quad 0<\theta<\varepsilon, \quad t=0$$

It may be pointed out here that the nature of $\phi(r, \theta, z, t)$ in (2.1) and $E(r, \theta, z)$ in (2.6) is so chosen that relevant transforms of these functions taken during the solution of the problem exist.

3. Solution of the problem

To solve the above boundary value problem we multiply both sides of (2.1) by sin $(m\pi\theta/\varepsilon)$ and integrate with respect to θ between the limits $(0, \varepsilon)$, Now making use of the result [11, p.427(8-1-12)] and boundary condition (2.5), we get

$$(3.1) \qquad \frac{\partial u_s}{\partial t} = \beta\left(\frac{\partial^2 u_s}{\partial r^2} + \frac{1}{r}\frac{\partial u_s}{\partial r} - \frac{p^2}{r^2}u_s + \frac{\partial^2 u_s}{\partial z^2}\right)$$

where $p=(m\pi/\varepsilon)$, $u_s \equiv u_s(r, m, z, t) = \int_0^{\varepsilon} u(r, \theta, z, t) \sin(m\pi\theta/\varepsilon)\, d\theta$ and similar interpretation for $\phi_s(r, m, z, t)$.

Again multiplying both sides of (3.1) by $rS_p(v_n r)$, integrating with respect to r from a to b, we get in view of a known result [7, p. 161(14)] and the boundary conditions (2.3) and (2.4), the following differential equation

$$(3.2)\qquad \frac{\partial \bar{u}_s}{\partial t}=\beta A(n)-\beta v_n^2 \bar{u}_s+\beta \frac{\partial^2 \bar{u}_s}{\partial z^2}+\bar{\phi}_s\ (n,\ m,\ z,\ t)$$

where $\bar{u}_s\equiv\bar{u}_s\ (n,\ m,\ z,\ t)=\int_a^b \bar{u}_s(r,\ m,\ z,\ t)r\ S_p(v_n r)\ dr$ and similar meaning for $\bar{\phi}_s(n,\ m,\ z,\ t)$. Also

$$(3.3)\qquad A(n)=\frac{\varepsilon}{m\pi}\,[1+(-1)^{m+1}]\left[\frac{b}{k_2}S_p(v_n b)u_2-\frac{a}{k_1}S_p(v_n a)u_1\right]$$

$S_p(v_n r)\equiv S_p(k_1,\ k_2,\ v_n r)$, and v_n is chosen as a positive root of the equation

$$(3.4)\qquad J_p(k_1,\ v_n a)\ G_p(k_2,\ v_n b)-J_p(k_2,\ v_n b)\ G_p(k_1,\ v_n a)=0$$

J_p and G_p are Bessel functions of first and second kinds respectively of order p.

To transform (3.2) into an ordinary differential equation, we multiply both sides of (3.2) by $(1/\sqrt{2\pi})\ e^{iwz}$, integrate with respect to z, between the limits $-\infty$ to ∞, make use of known results [11, p. 40(2-3-12), p. 517(F1)], subject to boundary condition (2.2), and obtain

$$(3.5)\qquad \frac{d\bar{u}_{s,f}}{dt}+B(n,\ w)\bar{u}_{s,f}=\beta\sqrt{2\pi}\ A(n)\ \delta(w)+\bar{\phi}_{s,f}\ (n,\ m,\ w,\ t)$$

where $\bar{u}_{s,f}\equiv\bar{u}_{s,f}(n,\ m,\ w,\ t)=(1/\sqrt{2\pi})\int_{-\infty}^{\infty}\bar{u}_s(n,\ m,\ z,\ t)e^{iwz}\ dz$ and

$$(3.6)\qquad B(n,\ w)=\beta(v_n^2+w^2)$$

Now making use of the well known Laplace transform and taking in view of the initial condition (2.6), the equation (3.5) reduces to

$$(3.7)\quad L\{\bar{u}_{s,f}\,;\,q\}=\frac{1}{q+B(n,\ w)}\left[\bar{E}_{s,f}(n,\ m,\ w)+L\{\bar{\phi}_{s,f}\,;\,q\}\right]+\frac{\beta\sqrt{2\pi}\ A(n)\ \delta(w)}{q\{q+B(n,\ w)\}}$$

where $L\{\bar{u}_{s,f}\,;\,q\}=\int_0^{\infty}e^{-qt}\ \bar{u}_{s,f}(n,\ m,\ w,\ t)\ dt$ and

$$(3.8)\quad \bar{E}_{s,f}(n,m,w)=(1/\sqrt{2\pi})\int_{-\infty}^{\infty}\int_0^{\varepsilon}\int_a^b rE(r,\theta,z)\ S_p(v_n r)\ \sin(m\pi\,\theta/\varepsilon)e^{iwz}dr\ d\theta\ dz$$

Further taking inverse Laplace transform in (3.7) and using the convolutiion property for this transform [11, p. 169(3-9-1)], we get

$$(3.9)\quad \bar{u}_{s,f}(n,m,w,t)=\bar{E}_{s,f}(n,m,w)\ e^{-B(n,w)t}+\int_0^t \bar{\phi}_{s,f}(n,m,w,T)e^{-B(n,w)(t-T)}dT$$
$$+\frac{\sqrt{2\pi}\beta\ A(n)\ \delta(w)}{B(n,w)}\ (1-e^{-B(n,w)t})$$

Lastly applying inversion theorems for finite sine, extended finite Hankel and Fourier transforms [11, p. 426 (8-1-7) ; p. 37 (2-3-2) ; 7, p. 160(9)], we obtain the following solution of the boundary value problem, by virtue of a known result [11, p. 486 (9-1-7)]

$$(3.10)\quad u(r,\theta,z,t)=(1/\alpha)\sqrt{(2/\pi)}\sum_{m=1}^{\infty}\sum_{n}(1/C_n)\ \sin\ (m\pi\theta/\varepsilon)\ S_p(v_n r)$$
$$\cdot\left[\left\{\int_{-\infty}^{\infty}\left(\bar{E}_{s,f}(n,m,w)\ e^{-B(n,w)t}+\int_0^t\bar{\phi}_{s,f}(n,m,w,T)\ e^{-B(n,w)(t-T)}\ dT\right)\right.\right.$$
$$\left.\left. e^{-iwz}\ dw\right\}+\frac{A(n)}{v_n^2}(1-e^{-v_n^2 t})\right]$$

where the summation (over n) being taken over the positive roots of (3,4) and the value of C_n is same as given in the paper due to Marchi and Zgrablich [7, p. 160(11)]. Also $A(n)$ and $B(n,w)$ are given by (3.3) and (3.6) respectively.

4. Heat conduction in a truncated wedge of infinite height with a moving source of heat within it

In this section, as a special case of the problem (2.1), we find the distribution of temperature in the wedge, having a moving source of heat within it. The boundary conditions are the same as stated with (2.1). The position of such source at any instant t is given by $z=vt$. Thus we take in (2.1), $\phi(r,\theta,z,t)=F(r,\theta)\ \delta(z-vt)$, wherein δ stands for the well known Dirac-delta function. Now applying a known result [11, p. 486(9-1-8)] and evaluating the T-integral, we find that the integral

$$\text{(i)}\quad \int_{-\infty}^{\infty}\left[\int_0^t \bar{\phi}_{s,f}(n,\ m,\ w,\ T)\ e^{-B(n,w)(t-T)}\ dT\right]\ e^{-iwz}\,dw$$

occurring in (3.10), reduces to the following expression

$$\text{(ii)}\qquad (1/\sqrt{2\pi})\ \beta^{-1}\ \bar{F}_s(n,m)\ I(m,n)$$

where

(4.1) $$\overline{F}_s(n,m)=\int_0^{\varepsilon}\int_a^{b} rF(r,\theta)\ S_p(v_n r)\ \sin\ (m\pi\theta/\varepsilon)\ dr\ d\theta$$

(4.2) $$I(m,n)=\int_{-\infty}^{\infty}\frac{e^{ivwt}-e^{-\beta(v_n^2+w^2)t}}{\left(w+\frac{iv}{2\beta}\right)^2+d^2}e^{-iwz}\ dw$$

and

(4.3) $$d^2=v_n^2+\frac{v^2}{4\beta^2}$$

Now, to evaluate the integral (4.2), we put $w+\frac{iv}{2\beta}=y$, break the integral, thus obtained, into four integals and obtain

(4.4) $$I(m,n)=e^{-v(z-vt)/2\beta}\int_{-\infty}^{\infty}\frac{e^{ivty}}{y^2+d^2}\ e^{-iyz}\ dy-e^{-\beta v_n^2 t-\frac{vz}{2\beta}}\Big[\int_{-\infty}^{\infty}\frac{e^{-\beta y^2 t}}{y^2+d^2}e^{-iyz}dy.$$
$$+\int_{-\infty}^{\infty}\frac{e^{ivty}}{y^2+d^2}e^{-iyz}\ dy+e^{v^2t/4\beta^2}\int_{-\infty}^{\infty}\frac{e^{-iyz}}{y^2+d^2}\ dy\Big]$$

Again, using the known results [4, p. 117 (3) & (7) ; p. 118 (1) ; p. 15(15)] to evaluate various integrals in the right hand side of (4.4), we get

(4.5) $$I(m,n)=\pi d^{-1}\Big[e^{-\{v(z-vt)/2\beta\}-d|z-vt|}-e^{-\beta v_n^2 t-\frac{vz}{2\beta}}\Big\{2^{-1}e^{d^2\beta t}\cdot(e^{-dz}\ \text{Erfc}\ \overline{d\sqrt{\beta t}-z/2\sqrt{\beta t}}+e^{dz}\ \text{Erfc}\ \overline{d\sqrt{\beta t}+z/2\sqrt{\beta t}})+e^{-d|z-vt|}+e^{(v^2t/4\beta^2)-d|z|}\Big\}\Big]$$

Thus by using (3.10), (i) and (ii), we get the following solution of thus modified boundary value problem

(4.6) $$u(r,\theta,z,t)=\alpha^{-1}\sqrt{(2/\pi)}\sum_{m=1}^{\infty}\sum_{n}(1/C_n)\ \sin\ (m\pi\theta/\varepsilon)\ S_p(v_n r)\Big[e^{-B(n,w)t}\int_{-\infty}^{\infty}\overline{E}_{s,f}(n,m,w)\ e^{-iwz}\ dw+(2\pi)^{-1/2}\beta^{-1}\ \overline{F}_s(n,m)\ I(m,n)+\frac{A(n)}{v_n^2}\ (1-e^{-\beta v_n^2 t})\Big]$$

where $\overline{F}_s(n,m)$, $I(m,n)$, $B(n,w)$ and $A(n)$ are given by (4.1), (4.5), (3.6) and (3.3) respectively.

5. An interesting special case of (2.1)

If in (2.1), we take $\phi(r,\ \theta,\ z,\ t)=MG(r)\delta(t)$, and replace the boundary condition (2.6) by $E(r,\ \theta,\ z)=NH(r)$ (where M and N are constants), we get the following solution of the corresponding boundary value problem, by virtue of a known result [11, p. 486 (9-1-7)]

$$(5.1)\qquad u(r,\theta,z,t)=\alpha^{-1}\sqrt{(2/\pi)}\sum_{m=1}^{\infty}\sum_{n}(1/C_n)\sin(m\pi\theta/\varepsilon)\,S_p(v_n r)$$
$$\times\left[p\{M\bar{G}(n)+N\bar{H}(n)\}e^{-\beta v_n^2 t}+\frac{A(n)}{v_n^2}(1-e^{-\beta v_n^2 t})\right]$$

where $P=\sqrt{(2/\pi)}\,\dfrac{\varepsilon}{m}[1+(-1)^{m+1}]$

$\bar{G}(n)=\int_a^b rG(r)\,S_p(v_n r)dr$ similar meaning for $\bar{H}(n)$ and $A(n)$ is given by (3.3).

6. Example

If in (2.1), we take source of heat

$$\phi(r,\theta,z,t)=r^{Q-1}\,\theta^{M-1}\,(\varepsilon^2-\theta^2)^{-N/2}P_q^N(\theta/\varepsilon)e^{-c|z|}\,\delta(t);$$

and further choose the function

$$E(r,\theta,z)=r^{Q-1}\,\theta^{d-1}(\varepsilon-\theta)^{g-1}\,e^{-h^2z^2}$$

and evaluate the various integrals occurring in (3.10) with the help of known results [11, p. 517 (F11) & (F8) ; 4, p. 15 (11) & (15) ; 1, p. 314 (7) ; 3, p. 90 (7) ; 10, p. 115(11) ; p. 18(1)], we get the following solution of the boundary value problem, after a little simplification

$$(6.1)\qquad u(r,\theta,z,t)=\pi\,\alpha^{-1}\sum_{m=1}^{\infty}\sum_{n}(1/C_n)\sin(m\pi\theta/\varepsilon)\,S_p(v_n r)\cdot\Big[mR(n)\,e^{-\beta v_n^2 t}$$
$$\Big\{2\varepsilon^{d+g-1}(1+4h^2\beta t)^{-1}e^{-(h^2z^2/1+4h^2\beta t)}\frac{\Gamma(d+1)\,\Gamma(g)}{\Gamma(d+g+1)}\cdot{}_2F_3\Big(\frac{d+1}{2},$$
$$\frac{d+2}{2};\frac{3}{2};\frac{d+g+1}{2},\frac{d+g+2}{2};-\frac{m^2\pi^2}{4}\Big)+\sqrt{\pi}\,\varepsilon^{M-N}e^{c^2\beta t}\Big(e^{-cz}\cdot$$
$$\cdot\mathrm{Erfc}\,\overline{c\sqrt{\beta t}-z/2\sqrt{\beta t}}+e^{cz}\quad\mathrm{Eafc}\,\overline{c\sqrt{\beta t}+z/2\sqrt{\beta t}}\Big)$$
$$\frac{2^{N-M-1}\,\Gamma(M+1)}{\Gamma\Big(\frac{2+M-N-q}{2}\Big)\Gamma\Big(\frac{3+M-N+q}{2}\Big)}\cdot{}_2F_3\Big(\frac{M+1}{2},\frac{M+2}{2};\frac{3}{2};$$
$$\frac{2+M-N-q}{2},\frac{3+M-N+q}{2};-\frac{m^2\pi^2}{4}\Big\}+\frac{A(n)}{v_n^2}(1-e^{-\beta v_n^2 t}\Big]$$

where

(6.2) $R(n)=S(n)\ (v_n)^{-Q}\Big[x\{(p+Q-1)\,J_p(v_nx)\ S_{Q-1,p-1}(v_nx)-J_{p-1}(v_nx)S_{Q,p}(v_nx)\}\Big]_a^b$

(6.3) $S(n)=G_p(k_1, v_na)+G_p(k_2,\ v_nb)-\frac{1}{2}\ \mathrm{cosec}\ p\pi\ [J_p(k_1,\ v_na)+J_p(k_2,\ v_nb)]\cdot[(-1)^p-e^{-ip\pi}]$

(6.4) $G_p(k_i,\ v_nx)=G_p(v_nx)+k_iv_nG_p(v_nx)\ \ (i=1,2)$

(6.5) $G_p(v_nx)=\frac{1}{2}\ \mathrm{cosec}\ p\pi\ [J_{-p}(v_nx)-e^{-ip}\ J_p(v_nx)]$

$S_{Q,p}(x)$ is the well known Lommel's function [3, p. 40 (71)], $A(n)$ is given by (3.3) and $p=(m\pi/\varepsilon)$.

Also in (6.1), ${}_2F_3(x)$ is the generalized hypergeometric function [10, p. 73 (1)] and $P_q^N(x)$ is the well known associated Legendre function.

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