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NUMERICAL RANGE THEORY FOR PSEUDO-BANACH ALGEBRAS

By T. Husain and A. Srinivasan

0. Introduction

In [7] Giles and the first author extended the concept of numerical range of an element of a normed algebra to locally multiplicatively convex algebra and studied some of the basic properties. In [8] Giles and Koehler studied it further. In this paper we study the numerical range of an element of a pseudo-Banach algebra introduced by Allan etc. in [3]. Among other results, we show that for any element of a pseudo-Banach algebra the numerical radius is equal to its spectral radius. Also its spectrum is compact and is contained in its numerical range which is a convex compact set. Hence we show that the convex hull of the spectrum of an element coincides with its numerical range. Moreover, we give a characterization of dissipative elements of a pseudo-Banach algebra. (The case for Banach algebras is known in [5]).

It is known [3] that every commutative Banach algebra with identity is a pseudo-Banach algebra and there are pseudo-Banach algebras which are not Banach algebras. It is interesting to note that in the case of a Banach algebra, the numerical radius is less than or equal to the spectral radius whereas in the present case of pseudo-Banach algebra, they are equal.

1. Preliminaries

The concept of pseudu-Banach algebra was introduced by Allan, Dale and McClure [3]. We reproduce the definition here:

DEFINITION 1.1. (Allan, Dales and McClure [3]) Let A be a commutative topological algebra over the complex field C with identity 1. A bound structure for A is a non-empty collection β of subsets of A satisfying the following conditions:

(i) Each $B_\alpha \in \beta$ is absolutely convex, bounded, $B_\alpha^2 \subset B_\alpha$ and $1 \in B_\alpha$;

(ii) Given B_1, $B_2 \in \beta$, there exists B_3 in β and $\lambda > 0$ such that $B_1 \cup B_2 \subset \lambda B_3$.

The pair (A, β) is called a *Bound algebra*.

For B_α in β, let $A(B_\alpha)=\{\lambda b : \lambda\in C,\ b\in B_\alpha\}$. In view of (i), $A(B_\alpha)$ is a subalgebra of A generated by B_α. The Minkowski functional of B_α defines a submultiplicative seminorm $\|\cdot\|_{B_\alpha}$ on $A(B_\alpha)$. If each $\|\cdot\|_{B_\alpha}$ is a norm, and if $A(B_\alpha)$ is a Banach algebra with respect to $\|\cdot\|_{B_\alpha}$, and if $A=\cup\{A(B_\alpha) ; B_\alpha\in\beta\}$, then A is called a *pseudo Banach algebra.*

If A is an algebra with identity 1, then $G(A)$ denotes the set of all invertible elements in A, A' its topological dual and A^* its algebraic dual. We follow the notion and terminologies of [3] and [5].

PROPOSITION 1.2. (Allan etc. [3]) *An algebra A is a pseudo-Banach algebra with respect to some bound structure, if and only if A is isomorphic with the inductive limit of an inductive system $(A_\alpha ; i_{\beta\alpha} : \alpha,\ \beta\in\Lambda,\ \alpha\leq\beta)$, of Banach algebras with identity and continuous unital monomorphisms.*

REMARK. Observe that a priori a pseudo-Banach algebra does not carry the inductive limit topology which is complete by definition. Its initial topology is in general coarser than the inductive limit topology. We assume that A is a complete topological algebra as well as a pseudo-Banach algebra in the sequel. The following simple known result is given here for the use in the sequel

PROPOSITION 1.3. *Let (A,β) be a pseudo-Banach algebra with the inductive limit topology. Then a linear functional f on (A,β) is continuous, if and only if for each α, $f|A_\alpha=f_\alpha$ is a continuous linear functional on the Banach algebra $(A_\alpha, \|\cdot\|_\alpha)$.*

2. Numerical range of an element of a Pseudo-Banach Algebra

DEFINITION 2.1. Let (A,β) be a pseudo-Banach algebra with identity 1. Recall $\beta=\{B_\alpha ; \alpha\in\Lambda\}$, where each B_α is an absolutely convex bounded set satisfying the conditions in 1.1. We put $A_\alpha=A(B_\alpha)$ which is a Banach algebra. We define $D(A,\beta ; 1)=\{f\in A' : f(1)=1,\ \|f|_{A_\alpha}\|\leq 1,$ for all $\alpha\in\Lambda\}$, and $D_\alpha(A, B_\alpha ; 1)=\{f\in A^* ; f|A_\alpha\in D\ (A_\alpha, \|\cdot\|_\alpha ; 1)\}$, where $D(A_\alpha, \|\cdot\|_\alpha : 1)=\{f_\alpha\in A'_\alpha : \|f_\alpha\|_\alpha=1=f_\alpha(1)\}$. Observe that for each $f_\alpha\in A_\alpha'$, by the Hahn-Banach theorem there exists a $g\in A^*$ such that $g|A_\alpha=f_\alpha$.

THEOREM 2.2. *Let (A,β) be a pseudo-Banach algebra with identity* 1 *and with the inductive limit topology. Then* $\bigcap_{\alpha\in\Lambda} D_\alpha(A, B_\alpha ; 1)=D(A,\ \beta ; 1)$.

PROOF. If $f \in \bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha ; 1)$, then f is a linear functional on A such that $f|A_\alpha = f_\alpha$ is a continuous linear functional on A_α, $\|f_\alpha\|_\alpha \leq 1$ and $f_\alpha(1)=1$ for each α in Λ. The continuity of f_α on A_α for all α in Λ implies the continuity of f on A by proposition 1.3. Clearly $f(1)=1$ and $\|f_\alpha\|_\alpha \leq 1$, for all α in Λ imply that $f \in D(A, \beta ; 1)$. Conversely, if $f \in D(A, \beta ; 1)$, then clearly $f_\alpha = f|A_\alpha \in A_\alpha'$, $f_\alpha(1)=1$ and $\|f_\alpha\|_\alpha \leq 1$, for each $\alpha \in \Lambda$. Thus we have:

$$\bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha ; 1) = D(A, \beta ; 1).$$

THEOREM 2.3. *Let (A, β) be a pseudo-Banach algebra with identity* 1 *endowed with the inductive limit topology. Then $D(A, \beta ; 1) = \varprojlim D_\alpha(A, B_\alpha ; 1)$ (projective limit, see* [5]*).*

PROOF. First we show that for α, β in Λ, $\alpha \leq \beta$ implies $D_\alpha(A, B_\alpha ; 1) \supset D_\beta(A, B_\beta ; 1)$. If $f \in D_\beta(A, B_\beta ; 1)$, then $f \in A^*$ and $f_\beta = f|B_\beta$ is in $D(A_\beta, \|\cdot\|_\beta ; 1)$. So $f_\beta \in A_\beta'$. Since $\alpha \leq \beta$ means $B_\alpha \subset B_\beta$, it follows that $f|A_\alpha = f_\beta|A_\alpha \in A_\alpha'$. This proves that $f \in D_\alpha(A, B_\alpha ; 1)$.

Now $\{D_\alpha(A, B_\alpha ; 1)\}_\Lambda$ is a family of subsets of A^* indexed by the directed set Λ such that for $\alpha \leq \beta$, $\alpha, \beta \in \Lambda$, $D_\alpha(A, B_\alpha ; 1) \supset D_\beta(A, B_\beta ; 1)$. For each α in Λ, the norm topology on $D_\alpha(A, B_\alpha ; 1)$ induced by $\|\cdot\|_\alpha$ is finer than the norm topology on $D_\beta(A, B_\beta ; 1)$ by $\|\cdot\|_\beta$, whenever $\alpha \leq \beta$ because $B_\alpha \subset B_\beta$ *for* $\alpha \leq \beta$. Take $i_{\alpha\beta}$ to be the canonical injection: $D_\beta(A, B_\beta ; 1) \rightarrow D_\alpha(A, B_\alpha ; 1)$ for $\alpha \leq \beta$, then $\varprojlim D_\alpha(A, B_\alpha ; 1)$ may be identified canonically with $\bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha ; 1)$ (See Bourbaki [6] page 50). But by Theorem 2.2, $\bigcap_{\alpha \in \Lambda} D_\alpha(A, B_\alpha ; 1) = D(A, \beta ; 1)$ and the result follows.

DEFINITION 2.4. Let (A, β) be a pseudo-Banach algebra and a an element of A. Since $A = \bigcup_\alpha \{A(B_\alpha) ; B_\alpha \in \beta\}$, $a \in A_\alpha$ for some α. Put $V_\alpha(A, B_\alpha ; a) = \{f(a) ; f \in D_\alpha(A, B_\alpha ; 1)\}$ and define the *numerical range of a* to be $V(A, \beta ; a) = \{f(a) ; f \in D(A, \beta ; 1)\}$.
$v(A, \beta ; a) = \sup\{|\lambda| ; \lambda \in V(A, \beta, a)\}$ is called the *numerical radius of a.*

REMARK. Since a complete locally convex algebra in which every element is bounded is a pseudo-Banach algebra (cf: [3]), the results of this chapter hold good for those locally convex algebras as well.

THEOREM 2.5. *Let (A, β) be a pseudo-Banach algebra and a an element of*

A. If $a \in A_\alpha$ *for some* α, *then* $V_\alpha(A, B_\alpha ; a) = V(A_\alpha, \|\cdot\|_\alpha ; a)$ *and* $V(A, \beta ; a) = \bigcap_\alpha \{V(A_\alpha, \|\cdot\|_\alpha ; a) ; a \in A_\alpha\}$. *Moreover,* $v(A, \beta ; a) = \inf\{v(A_\alpha, \|\cdot\|_\alpha ; a) \leq \|a\|_\alpha,\ a \in A_\alpha\}$.

PROOF. By definition,

$$\begin{aligned} V_\alpha(A, B_\alpha ; a) &= \{f(a) ; f \in D_\alpha(A, B_\alpha : 1)\} \\ &= \{g(a) ; g \in A^*,\ g_\alpha \in D(A_\alpha, \|\cdot\|_\alpha ; 1)\} \\ &= \{g_\alpha(a) ; g_\alpha \in D(A_\alpha, \|\cdot\|_\alpha ; 1)\} \\ &\quad \text{(Since } g(a) = g_\alpha(a), \text{ for } a \in A_\alpha) \\ &= V(A_\alpha, \|\cdot\|_\alpha ; a). \end{aligned}$$

Thus, by Theorem 2.2,

$$\begin{aligned} V(A, \beta ; a) &= \{f(a) ; f \in D(A, \beta ; 1)\} \\ &= \{g(a) ; g \in \bigcap_{\alpha \in A} D_\alpha(A, B_\alpha : 1)\} \\ &= \bigcap_\alpha \{g(a) ; g \in D_\alpha(A, B_\alpha : 1),\ a \in A_\alpha\} \\ &= \bigcap_\alpha \{V_\alpha(A, B_\alpha ; a) ; a \in A_\alpha\} \\ &= \bigcap_\alpha \{V(A_\alpha, \|\cdot\|_\alpha ; a) ; a \in A_\alpha\}. \end{aligned}$$

Finally, since $V(A, \beta ; a) = \bigcap_\alpha V(A_\alpha, \|\cdot\|_\alpha ; a)$, we have,

$$v(A, \beta ; a) = \inf_\alpha \{v(A_\alpha, \|\cdot\|_\alpha ; a) \leq \|a\|_\alpha,\ a \in A_\alpha\}.$$

THEOREM 2.6. *Let A be a pseudo-Banach algebra with identity* 1 *and a an element of A. Then,*

(i) $\mathrm{Sp}(A ; a) = \bigcap_\alpha \{\mathrm{Sp}(A_\alpha ; a) ; a \in A_\alpha\}$, *where* $\mathrm{Sp}(A_\alpha, a)$ *is the spectrum of* $a \in A_\alpha$.

(ii) $r(A ; a) = \inf_\alpha \{r(A_\alpha ; a) ; a \in A_\alpha\}$, *where* $r(A ; a) = \sup \{|\lambda| : \lambda \in \mathrm{Sp}(A ; a)\}$ *is the spectral radius.*

PROOF. (i) By definition, $\mathrm{Sp}(A ; a) = \{\lambda \in C : (\lambda 1 - a) \notin G(A)\}$. If $\lambda \in \mathrm{Sp}(A ; a)$ then $(\lambda - a) \notin G(A)$. We claim that $(\lambda - a) \notin G(A_\alpha)$ for each A_α for which $a \in A_\alpha$. For, otherwise, $(\lambda - a) \in G(A_\alpha)$ for some A_α for which $a \in A_\alpha$. But this implies that $(\lambda - a) \in G(A)$ (because $G(A_\alpha) \subset G(A)$) contradicting that $(\lambda - a) \notin G(A)$. Hence $\mathrm{Sp}(A ; a) \subset \mathrm{Sp}(A_\alpha ; a)$, for all α's for which $a \in A_\alpha$; and so $\mathrm{Sp}(A ; a) \subset \bigcap_\alpha \{\mathrm{Sp}(A_\alpha ; a) ; a \in A_\alpha\}$. On the other hand, if $\lambda \in \bigcap_\alpha \{\mathrm{Sp}(A_\alpha ; a) ; a \in A_\alpha\}$, then $(\lambda-a)\notin G(A_\alpha)$ for all A_α for which $a\in A_\alpha$. But the algebra A being the union of the subalgebras A_α, which are outer directed by inclusion, we see that $(\lambda-a)\notin G(A)$; and therefore $\lambda\in \mathrm{Sp}(A ; a)$. This proves (i).

(ii) By definition and (i), $r(A ; a)=\sup\ \{|\lambda| ; \lambda\in \mathrm{Sp}(A, a)\}=\sup\ \{|\lambda| ; \lambda\in \bigcap_\alpha \mathrm{Sp}(A_\alpha ; a)\}=\inf_\alpha\{r(A_\alpha ; a) ; a\in A_\alpha=A(B_\alpha)\}$, because $\{B_\alpha\}$ is outer directed.

THEOREM 2.7. *Let (A,β) be a pseudo-Banach algebra and a an element of A. Then* $\mathrm{Sp}(A ; a)\subset V(A,\beta ; a)$.

PROOF. We know that $\mathrm{Sp}(A_\alpha ; a)$ is contained in $V(A_\alpha, \|\cdot\|_\alpha ; a)$ for $a\in A_\alpha$, because A_α is a Banach algebra ([5], page 19, Th. 6).

Hence by Theorem 2.6, $\mathrm{Sp}(A ; a)=\bigcap_\alpha \{\mathrm{Sp}(A_\alpha ; a) ; a\in A_\alpha\}$

$$\subset\bigcap_\alpha \{V(A_\alpha, \|\cdot\|_\alpha ; a) ; a\in A_\alpha\}$$

$$=V(A,\beta ; a) \text{ by Theorem 2.5.}$$

3. Some properties of the numerical range and spectrum

PROPOSITION 3.1. *Let (A,β) be a pseudo-Banach algebra and a an element of A. Then* $\mathrm{Sp}(A ; a)$ *is a compact subset of* **C** *and* $V(A,\beta ; a)$ *a convex compact subset of* **C**.

PROOF. By Theorem 2.6, we have $\mathrm{Sp}(A, a)=\bigcap_\alpha \{\mathrm{Sp}(A_\alpha ; a) ; a\in A_\alpha\}$, where each $\mathrm{Sp}(A_\alpha ; a)$ is a nonempty compact subset of **C** [5]. Since $\mathrm{Sp}(A ; a)$ is the intersection of nonempty compact subsets of **C**, it is compact. Further, since by Theorem 2.5

$V(A,\beta ; a)=\bigcap_\alpha \{V(A_\alpha, \|\cdot\|_\alpha ; a) ; a\in A_\alpha\}$, where each $V(A_\alpha, \|\cdot\|_\alpha ; a) ; a\in A_\alpha$ is a convex compact subset of **C**, by Th. 3 (page 16, [5]) it follows that $V(A, \beta ; a)$ is a convex compact subset of **C**.

THEOREM 3.2. *Let (A,β) be a pseudo-Banach algebra with the inductive limit topology and a an element of A. Then,* $r(A ; a)=\inf\ \{r(A_\alpha ; a) ; a\in A_\alpha\}=\inf_\alpha\{\|a\|_\alpha, a\in A_\alpha\}$ *and* $r(A ; a)=v(A,\beta ; a)$.

PROOF. Let $\beta(a)=\inf\ \{\|a\|_\alpha, B_\alpha\in\beta ; a\in A_\alpha=A(B_\alpha)\}$. By Theorem 2.6 (ii),

$$r(A ; a)=\inf\ \{r(A_\alpha ; a) ; a\in A_\alpha\}$$

$$=\inf_\alpha\{\sup\ \{|\lambda| ; \lambda\in Sp(A ; a)\}\}\leq\inf_\alpha\{\|a\|_\alpha ; a\in A_\alpha\}$$

$$=\beta(a).$$

Now we prove that $\beta(a)\leq r(A;a)$. Suppose $a\in A_\alpha=A(B_\alpha)$ for some $B_\alpha\in\beta$. If $z\notin \mathrm{Sp}(A_\alpha;a)$ and $|z|>\|a\|_\alpha$, then $(z-a)^{-1}\in A_\alpha$ and $(z-a)^{-1}=z^{-1}+z^{-2}a+z^{-3}a^2+\cdots$, in which the series converges absolutely in A_α. Thus if $f\in A'$ and $g(z)=f((z-a)^{-1})$, then $g(z)=z^{-1}f(1)+z^{-2}f(a)+z^{-3}f(a^2)+\cdots(*)$, for $z\notin \mathrm{Sp}(A_\alpha;a)$ and $|z|>\|a\|_\alpha\geq\beta(a)$. Clearly g is holomorphic for $|z|>r(A_\alpha;a)$ and hence from its Laurent expansion (*) we have $\limsup_{n\to\infty}|f(a^n)|^{1/n}\leq\limsup_{n\to\infty}\sup_\alpha\|a^n\|_\alpha^{1/n}$. Thus $\limsup_{n\to\infty}|f(a^n)|^{1/n}\leq r(A;a)$, $(f\in A')$ proves that $\beta(a)\leq r(A;a)$. But then $\beta(a)\leq r(A;a)=\inf_\alpha r(A_\alpha;a)\leq\inf_\alpha v(A_\alpha;\|\cdot\|_\alpha;a)\leq\inf_\alpha\{\|a\|_\alpha;a\in A_\alpha\}=\beta(a)$, which also implies that $\inf_\alpha r(A_\alpha;a)=\inf_\alpha v(A,B_\alpha,a)$. Hence we have $r(A;a)=v(A,\beta;a)=\inf_\alpha\{\|a\|_\alpha,a\in A_\alpha\}=\beta(a)$.

THEOREM 3.3. *Let (A,β) be a pseudo-Banach algebra and a an element of A. Then the closure of the convex hull of* Sp (A,a), *denoted by* Co Sp$(A;a)$, *coincides with $V(A,\beta;a)$.*

PROOF. By Theorem 2.7 and Prop. 3.1, $\mathrm{Sp}(A;a)$ is a compact subset of $\boldsymbol{C}$ and $V(A,\beta;a)$ is a convex compact subset of $\boldsymbol{C}$ such that $\mathrm{Sp}(A;a)\subseteq V(A,\beta;a)$. Also by Theorem 3.2, $r(A;a)=v(A,\beta;a)$. Hence Co $\mathrm{Sp}(A;a)=V(a,\beta;a)$, because the latter is a compact convex subset.

THEOREM 3.4. *Let F be a closed (under the inductive limit topology) subalgebra of the pseudo-Banach algebra (A,β). Let (F,β') denote F with the bound structure β' restricted to F and let a be an element of F. Then $V(A,\beta;a)=V(F,\beta';a)$.*

PROOF. For each α we have $A(B_\alpha)\supset F(B_\alpha')$ i.e $A_\alpha\supset F_\alpha$, where $B_\alpha'=F\cap B_\alpha$. Also by the Banach-algebra numerical range theory (page 16, Th. 4, [5]) for each α when $a\in F_\alpha$, we have: $V(A_\alpha,\|\cdot\|_\alpha;a)=V(F_\alpha,\|\cdot\|_\alpha;a)$ or $V_\alpha(A,B_\alpha;a)=V_\alpha(F,B_\alpha';a)$, for $a\in F_\alpha$.

Hence $V(A,B;a)=\bigcap_\alpha\{V_\alpha(A,B_\alpha;a);a\in A_\alpha\}$ (Theorem 2.5)

$$=\bigcap_\alpha\{V_\alpha(F,B_\alpha';a);a\in F_\alpha\}=V(F,\beta;a).$$

REMARK. The above theorem is not true in general for any topological algebra, e.g., see [9]. It is, however, true for Banach algebras [5].

COROLLARY 3.5. *Let (A,β) be a pseudo-Banach algebra and a an element of A. Then $V(A,\beta\,;a)=V(P(a),\beta\,;a)$, where $P(a)$ is the algebra of polynomials with complex coefficients.*

PROOF. This is an immediate consequence of Theorem 4 (page 16, [5]) and Theorem 3.4.

The following properties (Theorem 3.6) are known to be true for Banach algebras. We show them here for pseudo-Banach algebras.

THEOREM 3.6. *Let (A,β) be a pseudo-Banach algebra with the inductive limit topology, let a,b be elements of A and $p,q\in C$. Let $V(A,\beta;a)$ be denoted by $V(A\,;a)$ for convenience. Then the following properties hold:*

(i) $V(A,a+b)\subset V(A\,;a)+V(A\,;b)$,

(ii) $V(A\,;p+qa)=p+qV(A\,;a)$, and $v(A\,;p+qa)\leq|p|+|q|\;v(A\,;a)$,

(iii) $v(A\,;pa)=|p|v(A\,;a)$,

(iv) $v(A\,;a+b)\leq v(A\,;a)+v(A\,;b)$,

(v) $r(A\,;a+b)\leq r(A\,;a)+r(A\,;b)$,

(vi) $r(A\,;ab)\leq r(A\,;a)\;r(A\,;b)$ and $v(A\,;ab)\leq v(A\,;a)\;v(a\,;b)$,

(vii) $v(A\,;a^n)=v^n(A\,;a)$ and $r(A\,;a^n)=r^n(A\,;a)$.

PROOF. They are easy to verify. For the Banach algebra case, see [5].

THEOREM 3.7. *Let (A,β) be a pseudo-Banach algebra and a an element of A. Then*

$$\max \operatorname{Re} V(A\,;\beta\,;a)=\inf_{p>0} p^{-1}\ \{\inf_{\alpha}\|1+pa\|_{\alpha}-1\,;a\in A_{\alpha}\}$$

$$=\lim_{p\to0^+} p^{-1}\ \{\inf_{\alpha}\ \|1+pa\|_{\alpha}-1\,;a\in A_{\alpha}\}.$$

PROOF. By Theorem 2.5, we have,

$$\max \operatorname{Re} V(A,\beta\,;a)=\max \operatorname{Re} \bigcap_{\alpha}\ \{V(A_{\alpha},\|\cdot\|_{\alpha}\,;a)a\in A_{\alpha}\}.$$

$$=\inf_{\alpha}\ \{\max \operatorname{Re} V(A_{\alpha},\|\cdot\|_{\alpha}\,;a)\,;a\in A_{\alpha}\}$$

$$=\inf_{\alpha}\ \{\inf_{p>0} p^{-1}\ (\|1+pa\|_{\alpha}-1)\,;a\in A_{\alpha}\}\ \text{(by [5])}$$

or

$$\lim_{p\to0^+}$$

$$=\inf_{p>0} p^{-1}\ \{\inf_{\alpha}\ \|1+pa\|_{\alpha}-1),\ a\in A_{\alpha}\}.$$

$$\text{or} \quad \lim_{p\to 0^+}$$

THEOREM 3.8. *Let (A,β) be a pseudo-Banach algebra and a an element of A. Then the set-valued map: $a\to V(A,\beta\,;a)$ is upper semicontinuous.*

PROOF. Observe that $a\to V(A,\beta\,;a)$ is continuous if A is endowed with the inductive limit topology. Since for every $a\in A$, $V(A,\beta\,;a)$ is a convex compact subset of A by Theorem 3.1, the set-valued mapping: $a\to V(A,\beta\,;a)$ is upper semicontinuous as in (cf. [4]).

THEOREM 3.9. *Let (A,β) be a pseudo-Banach algebra and a an element of A. Then*

$$\max \operatorname{Re} V(A,\beta\,;a)=\sup_{p>0}\left[\frac{1}{p}\log\{\inf_{\alpha}\|\exp(pa)\|_{\alpha}\,;a\in A_{\alpha}\}\right].$$
$$\text{or} \quad \lim_{p\to 0^+}$$

PROOF. $\max \operatorname{Re} V(A,\beta\,;a)=\max \operatorname{Re} \bigcap_{\alpha}\{V(A_{\alpha},\|\cdot\|_{\alpha}\,;a)\,;a\in A_{\alpha}\}$

$$=\inf_{\alpha}\max \operatorname{Re}\{V(A_{\alpha},\|\cdot\|_{\alpha}\,;a)\,;a\in A_{\alpha}\}$$
$$=\inf_{\alpha}\sup_{p>0}[p^{-1}\log\{\|\exp(pa)\|_{\alpha}\,;a\in A_{\alpha}\}]$$
$$\text{or} \quad \lim_{p\to 0^+}$$
$$=\sup_{p>0}[p^{-1}\log(\inf_{\alpha}\{\|\exp(pa)\|_{\alpha}\,;a\in A_{\alpha}\})].$$
$$\text{or} \quad \lim_{p>0^+}$$

DEFINITION 3.10. An element of a pseudo-Banach algebra is said to be dissipative if Re $z\leq 0$, for all $z\in V(A,\beta\,;a)$. (See [5] for the Banach algebra case.)

THEOREM 3.11. *Let (A,β) be a pseudo-Banach algebra and a an element of A. Then a is dissipative if and only if* $\inf_{\alpha}\{\|\exp(ta)\|_{\alpha}\,;a\in A_{\alpha}\}\leq 1$, $(t>0)$.

PROOF. Applying Theorem 3.9, we see that a is dissipative if and only if

$\log \inf_{\alpha} \{\|\exp(ta)\|_{\alpha} ; a \in A_{\alpha}\} \leq 0$, i.e. if and only if $\inf_{\alpha} \{\|\exp(ta)\|_{\alpha} ; a \in A_{\alpha}\} \leq 1$, $(t>0)$.

THEOREM 3.12. *Let (A, β) be a pseudo-Banach algebra and a an element of A. Then,*

$$\max \operatorname{Re} \operatorname{Sp}(A ; a) = \begin{matrix}\inf_{p>0} \\ \text{or} \\ \lim_{p \to 0^+}\end{matrix} \left\{\frac{1}{p} \log(\inf_{\alpha} \{\|\exp(pa)\|_{\alpha} ; a \in A_{\alpha}\})\right\}.$$

PROOF. $\max \operatorname{Re} \operatorname{Sp}(A ; a) = \max \operatorname{Re} \bigcap_{\alpha} \{\operatorname{Sp}(A_{\alpha} ; a) : a \in A_{\alpha}\}$

$$= \max \operatorname{Re} \{z ; z \in \operatorname{Sp}(A_{\alpha} ; a) ; a \in A_{\alpha}\}$$

$$= \inf_{\alpha} \{\max \operatorname{Re} \operatorname{Sp}(A_{\alpha} ; a) ; a \in A_{\alpha}\}$$

$$= \inf_{\alpha} \left[\begin{matrix}\inf_{p>0} \\ \text{or} \\ \lim_{p \to 0^+}\end{matrix} \left\{\frac{1}{p} \log (\|\exp(pa)\|_{\alpha} ; a \in A_{\alpha})\right\}\right]$$

$$= \begin{matrix}\inf_{p>0} \\ \text{or} \\ \lim_{p \to 0^+}\end{matrix} \frac{1}{p} \log (\inf_{\alpha} \{\|\exp(pa)\|_{\alpha} ; a \in A_{\alpha}\}).$$

McMaster University
Hamilton, Ontario
Canada, L8S 4K1

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